One-Sided Discrepancy of Linear Hyperplanes in Finite Vectorspaces

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Abstract

In this paper we investigate the c–color discrepancy, $c \geq 2$ and the one-sided c–color discrepancy of linear hyperplanes in the finite vector space $\mathbb{F}_q^r$. We denote by $\mathcal{H}$ the hypergraph with vertex set $\mathbb{F}_q^r$ and all linear hyperplanes (i.e. subspaces of codimension 1) in $\mathbb{F}_q^r$ as hyperedges. We show that the discrepancy of $\mathcal{H}$ is $\Theta_q\left(\sqrt{nz(1-z)}\right)$, where $n = |\mathbb{F}_q^r|$ and $z = \frac{(q-1) \mod c}{c}$. With Fourier analysis on $\mathbb{F}_q^r$ we further prove that the one-sided discrepancy of $\mathcal{H}$ is bounded from below by $\Omega_q\left(\sqrt{\frac{nz(1-z)}{c}}\right)$ which differs from the upper bound by a factor of $\sqrt{c}$. For large c, i.e. $c \geq qn^{1/3}$ we close the gap proving a $\Theta_q\left(\sqrt{\frac{n}{c}}\right)$ behaviour of the one-sided discrepancy. All together this exhibits a new example for a hypergraph with (almost) tight discrepancy bounds.

1 Introduction

Basics about Discrepancy. Let V be a finite set and $\mathcal{E}$ a subset of 2^V. Then $\mathcal{H} := (V, \mathcal{E})$ is called a hypergraph. A c–coloring of $\mathcal{H}$ is a function $\chi : V \to M_c$, where M_c is a set of cardinality c. For convenience we take $M_c = \{1, 2, \ldots, c\} =: [c]$, but in applications a different choice of M_c can be helpful (see Babai et al., [BHK2001]).
Let $A_i := \chi^{-1}(i)$ be the color-class of the color $i \in [c]$ in V. The c–color discrepancy of $\mathcal{H}$ with respect to χ is defined by

$$\mathrm{disc}(\mathcal{H}, \chi, c) = \max_{i\in[c]} \max_{E\in\mathcal{E}} \left| |A_i \cap E| - \frac{|E|}{c} \right|$$

and the c–color discrepancy of $\mathcal{H}$ is

$$\mathrm{disc}(\mathcal{H}, c) = \min_{\chi:V\to[c]} \mathrm{disc}(\mathcal{H}, \chi, c).$$

For $c = 2$ the c–color discrepancy is exactly half of the common two-color discrepancy where the two colors are represented by 1 and -1. To go further in Discrepancy theory, one can

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consider [BS95] and [Mat99]. The c–color discrepancy measures deviations from the average value $\frac{|E|}{c}$ in both directions. Therefore one cannot decide whether the discrepancy is caused by a lack or an excess of vertices in a hyperedge in one color. To measure the guaranteed excess of vertices in one hyperedge and color, we define another discrepancy notion. The one-sided c–color discrepancy of $\mathcal{H}$ with respect to χ is defined by

$$\operatorname{disc}^{+}(\mathcal{H},\chi,c)=\max_{i\in[c]}\max_{E\in\mathcal{E}}\left(|A_i\cap E|-\frac{|E|}{c}\right)$$

and the one-sided c–color discrepancy of $\mathcal{H}$ is

$$\operatorname{disc}^{+}(\mathcal{H},c)=\min_{\chi:V\to[c]}\operatorname{disc}^{+}(\mathcal{H},\chi,c).$$

Trivially we have $\operatorname{disc}^{+}(\mathcal{H},c)\leq\operatorname{disc}(\mathcal{H},c)$. And since $\sum_{i\in[c]}\left(|A_i\cap E|-\frac{|E|}{c}\right)=0$ for every $E\in\mathcal{E}$, it also holds

$$\operatorname{disc}(\mathcal{H},c)\leq(c-1)\operatorname{disc}^{+}(\mathcal{H},c). \tag{1}$$

Let $\mathbb{F}_q$ be the field of q elements, where $q=p^k$ is a prime power and $V=\mathbb{F}_q^r$ is the r–dimensional vector space over $\mathbb{F}_q$. Let $\mathcal{E}$ be the set of linear hyperplanes of V. This means that $\mathcal{E}$ is the set of all subspaces of codimension 1. For a set $S\subseteq\mathbb{F}_q^r$ we define $S^{\sharp}:=S\setminus\{0\}$. With $n:=|V|=q^r$, $\mathcal{H}=(V,\mathcal{E})$ is an $\frac{n}{q}$–uniform hypergraph with n vertices and $|\mathcal{E}|=\frac{n-1}{q-1}$ hyperedges.

The Results. Let $z=\frac{(q-1)\ mod\ c}{c}$. For $z=0$, i.e. $c|(q-1)$, the c–color discrepancy and one-sided discrepancy both are exactly $\frac{c-1}{c}$. For $z\neq0$ we prove the upper bound for both, the c–color discrepancy and the one-sided c–color discrepancy:

$$\operatorname{disc}^{+}(\mathcal{H},c)\leq\operatorname{disc}(\mathcal{H},c)\leq\alpha\sqrt{nz(1-z)}\left(\frac{c}{n}\right)^{\frac{1}{2(r-1)}} \tag{2}$$

for a constant $\alpha>0$. Our lower bound for the c–color discrepancy is

$$\frac{1}{q}\sqrt{nz(1-z)}-1\leq\operatorname{disc}(\mathcal{H},c). \tag{3}$$

For $r\geq6$ the lower bound for the one-sided c–color discrepancy is

$$\frac{\sqrt{z(1-z)}}{4q(q-1)}\sqrt{\frac{n}{c}}-1\leq\operatorname{disc}^{+}(\mathcal{H},c). \tag{4}$$

For constant q, by (2) and (3) the c–color discrepancy is $\Theta(\sqrt{nz(1-z)})$ while the lower bound for the one-sided c–color discrepancy in (4) differs from the upper bound $O(\sqrt{nz(1-z)})$ in (2) by a factor of $\sqrt{\frac{1}{c}}$. We can close this gap in a special situation. In fact, for a large number of colors c, i.e. $c\geq qn^{1/3}$ and if $r\geq4$ we get the lower bound

$$\frac{1}{22\sqrt{q}}\sqrt{\frac{n}{c}}-1\leq\operatorname{disc}^{+}(\mathcal{H},c). \tag{5}$$

Thus, in the case $c \geq qn^{1/3}$ the one-sided c–color discrepancy is presicely of the order $\Theta_q(\sqrt{\frac{n}{c}})$.
All together we have a new hypergraph with tight discrepancy behaviour.
Methods. The methods in proving the bounds are of combinatorial as well as of analytic type. The bounds for the discrepancy function are obtaint with standard methods (eigenvalue calculation, bounded Vapnik-Červonenkis dimension).
The hard work is to prove the lower bound for the one-sided discrepancy function. There we invoke Fourier analysis on the additive group $\mathbb{F}_q^r$ and the pigeon hole principle at many times. First of all, we observe that the one-sided discrepancy $\mathrm{disc}^+(\mathcal{H}, \chi, c)$ for a c–coloring χ of V can be written as

$$\mathrm{disc}^+(\mathcal{H}, \chi, c) = \max_{i \in [c]} \max_{E \in \mathcal{E}} \left(\frac{\hat{A}_i(E^\perp)}{q} + \frac{|A_i|}{q} - \frac{n}{qc} \right), \tag{6}$$

where $\hat{A}_i$ is the Fourier transform of the indicator function $\mathbf{1}_{A_i}$ in $\mathbb{F}_q^r$. The exact definition is given in subsection 3.1. We prove that for any set $A \subseteq \mathbb{F}_q^r$ there exists a $E \in \mathcal{E}$ with $\hat{A}(E^\perp) \geq -1$, so $\hat{A}(.)$ the Fourier Transform cannot be too small. Now a certain tradeoff between the size of A_i and $\hat{A}_i(E^\perp)$ in (6) can be shown.

2 Discrepancy of $\mathcal{H}$

In this section we give both, a lower and an upper bound for the c–color discrepancy of the hypergraph $\mathcal{H}$. We can assume that the number of colors c does not exceed the number of vertices n. Otherwise the discrepancy is at most 1. Therefore the factor $\left(\frac{c}{n}\right)^{\frac{1}{2(r-1)}}$ is a decreasing function in the upper bound. First we consider the case $c|(q-1)$.

Proposition 2.1. *If $c|(q-1)$ we have*

$$\mathrm{disc}^+(\mathcal{H}, c) = \mathrm{disc}(\mathcal{H}, c) = \frac{c-1}{c}.$$

Proof. Let $\mathcal{E}_1$ denote the set of all one-dimensional subspaces of V. It holds

$$V = \{0\} \cup \bigcup_{W \in \mathcal{E}_1} (W \setminus \{0\}) \tag{7}$$

and this union is disjoint. For every $W \in \mathcal{E}_1$ the set $W \setminus \{0\}$ contains $q-1$ elements. Using $c|(q-1)$ we can color $W \setminus \{0\}$ in an exactly balanced way. We just color $\frac{q-1}{c}$ vertices of $W \setminus \{0\}$ in each color. Doing this for every $W \in \mathcal{E}_1$ and coloring the origin with the color 1 we get a c–coloring $\chi : V \to [c]$. For every linear hyperplane $E \in \mathcal{E}$ we denote by $\mathcal{E}_1(E)$ the set of all one-dimensional subspaces of E. Then (7) holds as well for E. For every $E \in \mathcal{E}$ and every $i \in [c]$ we have

$$\begin{aligned} \left||A_i \cap E| - \tfrac{1}{c}|E|\right| &= \left||A_i \cap \{0\}| - \tfrac{1}{c}|\{0\}| + \sum_{W \in \mathcal{E}_1} \left(|A_i \cap (W \setminus \{0\})| - \tfrac{1}{c}|(W \setminus \{0\})|\right)\right| \\ &= \left||A_i \cap \{0\}| - \tfrac{1}{c}\right| \\ &= \begin{cases} \frac{c-1}{c} & : \quad i = 1, \\ \frac{1}{c} & : \quad \textit{otherwise.} \end{cases} \end{aligned}$$

Thus we have

$$\operatorname{disc}^+(\mathcal{H},c) \leq \operatorname{disc}(\mathcal{H},c) \leq \frac{c-1}{c}.$$

Let $\chi : V \to [c]$ be a c–coloring and $E \in \mathcal{E}$ a hyperedge. Then there exists an $i \in [c]$ with $|A_i \cap E| \geq \left\lceil \frac{q^{r-1}}{c} \right\rceil = \frac{q^{r-1}-1}{c} + 1$. This yields

$$\operatorname{disc}(\mathcal{H},c,\chi) \geq \operatorname{disc}^+(\mathcal{H},c,\chi) \geq |E \cap A_i| - \tfrac{1}{c}|E| \geq \frac{q^{r-1}-1}{c} + 1 - \frac{q^{r-1}}{c} = \frac{c-1}{c}.$$

□

In the rest of the paper we assume that $c \nmid (q-1)$.

Theorem 2.2. *Let $z := \frac{(q-1) \mod c}{c}$. There exists a constant $\alpha > 0$ such that*

$$\frac{1}{q}\sqrt{nz(1-z)} - 1 \leq \operatorname{disc}(\mathcal{H},c) \leq \alpha \sqrt{nz(1-z)} \left(\frac{c}{n}\right)^{\frac{1}{2(r-1)}}.$$

2.1 The Lower Bound

Calculating a lower bound for the discrepancy of the hypergraph $\mathcal{H}$ becomes much easier, if we delete the origin from the vector space. Let us define $V' := V \setminus \{0\}$, $\mathcal{E}' := \{E \cap V' \mid E \in \mathcal{E}\}$, and $\mathcal{H}' := (V', \mathcal{E}')$. It is obvious that $\operatorname{disc}(\mathcal{H},c) \geq \operatorname{disc}(\mathcal{H}',c) - 1$, thus we can focus on $\mathcal{H}'$. The hypergraph $\mathcal{H}'$ has an interesting property. For two arbitrary elements $x, y \in V'$ let us denote by $d(x,y)$ the number of hyperedges $E \in \mathcal{E}'$ with $x, y \in E$ and call it the pair-degree of x and y. One can check that

$$d(x,y) := \begin{cases} \frac{q^{r-1}-1}{q-1} & , \text{ if } x \text{ and } y \text{ are linearly dependent,} \\ \frac{q^{r-2}-1}{q-1} & , \text{ otherwise.} \end{cases}$$

Let M be the incidence matrix of $\mathcal{H}'$: the rows of M correspond to the hyperedges of $\mathcal{H}'$, the columns to the vertices of $\mathcal{H}'$ and the component $M_{i,j}$ of M is 1 if $v_j \in E_i$ and is 0 otherwise. W.l.o.g. we can assume that the columns of M are arranged according to the one-dimensional subspaces of V. Let us denote by $\left(M^T M\right)_{x,y}$ the component of the matrix $M^T M$ in the row that corresponds to the vertex $x \in V'$ and the column that corresponds to the vertex $y \in V'$. Then we have $\left(M^T M\right)_{x,y} = d(x,y)$, and the matrix $M^T M$ is of the following form: on the diagonal there are $\frac{n-1}{q-1}$ copies of the $(q-1) \times (q-1)$–matrix with all components equal to $\frac{q^{r-1}-1}{q-1}$, and all other components of M are equal to $\frac{q^{r-2}-1}{q-1}$.
In this situation we can use the following lemma in which we extend Theorem 2.8 and Corollary 2.9 from [BS95] to block diagonal matrices and the c–color discrepancy. For simplicity we focus on block diagonal matrices consisting of blocks of same size and identical elements.

Lemma 2.3. *Let $\mathcal{H} = (V, \mathcal{E})$ be a finite hypergraph, set $m := |\mathcal{E}|$ and let M be the incidence matrix of $\mathcal{H}$. If for some block diagonal matrix D consisting of k copies of an $l \times l$–matrix*

X with all entries equal to $x > 0$, the matrix $M^TM - D$ is positive semidefinite, then with $z := \frac{l \mod c}{c}$

$$\operatorname{disc}(\mathcal{H}, c) \geq \left(\frac{kxz(1-z)}{m}\right)^{\frac{1}{2}}.$$

Proof. We follow the approach to c-color discrepancies in [DS2003]. The color $i \in [c]$ is described by a vector $m^{(i)} \in \mathbb{R}^c$ with

$$m_j^{(i)} := \begin{cases} \frac{c-1}{c} & : \quad i = j, \\ -\frac{1}{c} & : \quad \textit{otherwise}. \end{cases}$$

We define $M_c := \{m^{(i)} \mid i \in [c]\}$ and fix an optimal c–coloring $\chi : V \to M_c$ with respect to the c–color discrepancy. Furthermore, for all $i \in [l]$ we denote the coloring of the set V_i of vertices coresponding to the i–th block of D by $\chi_i : V_i \to M_c$. Then

$$\begin{aligned}
\operatorname{disc}(\mathcal{H}, c) &= \|(M \otimes I_c)\chi\|_\infty \\
&\geq \frac{1}{\sqrt{cm}}\|(M \otimes I_c)\chi\|_2 \\
&= \left(\frac{1}{cm}\chi^T(M \otimes I_c)^T(M \otimes I_c)\chi\right)^{\frac{1}{2}} \\
&= \left(\frac{1}{cm}\chi^T(M^TM \otimes I_c)\chi\right)^{\frac{1}{2}} \\
&= \left(\frac{1}{cm}\chi^T[(M^TM - D) \otimes I_c + D \otimes I_c]\chi\right)^{\frac{1}{2}} \\
&\geq \left(\frac{1}{cm}\chi^T(D \otimes I_c)\chi\right)^{\frac{1}{2}} \\
&= \left(\frac{1}{cm}\sum_{i \in [k]}\left(\chi_i^T(X \otimes I_c)\chi_i\right)\right)^{\frac{1}{2}} \\
&\geq \left(\frac{kx}{cm}\left(cz(1-z)^2 + c(1-z)z^2\right)\right)^{\frac{1}{2}} \\
&= \left(\frac{kxz(1-z)}{m}\right)^{\frac{1}{2}}.
\end{aligned}$$

□

Now we can apply Lemma 2.3. Here the block diagonal matrix D consists of $\frac{n-1}{q-1}$ block matrices on the diagonal. All of them are $(q-1) \times (q-1)$–matrices with all components equal to $\frac{q^{r-1}-1}{q-1} - \frac{q^{r-2}-1}{q-1} = q^{r-2}$. Hence the lower bound in Theorem 2.2 is established. Note that the term -1 is caused by the origin.

2.2 The Upper Bound

For the proof of the upper bound in Theorem 2.2 we invoke the VC-dimension of the hypergraph $\mathcal{H}$.

Definition 2.4. *Let $\mathcal{H} = (V, \mathcal{E})$ be a finite hypergraph and set $n := |V|$. The VC-dimension of $\mathcal{H}$ is defined by*

$$\dim(\mathcal{H}) := \max\{m \in [n] \mid \exists A \subseteq V : |A| = m,\ \mathcal{E} \cap A = 2^A\},$$

where $\mathcal{E} \cap A := \{E \cap A \mid E \in \mathcal{E}\}$.

Lemma 2.5. $\dim(\mathcal{H}) = r - 1$.

Proof. First of all each $A \subseteq V$ with the property $\mathcal{E} \cap A = 2^A$ is linearly independent. Furthermore a set A with $\mathcal{E} \cap A = 2^A$ and maximal cardinality cannot contain a complete basis, because a complete basis is not contained in any linear hyperplane. Thus the VC-dimension of $\mathcal{H}$ is at most $r - 1$.
On the other hand we show that every subset containing up to $r - 1$ linearly independent elements of V is shatered by $\mathcal{H}$. Let $\{v_1, v_2, \ldots, v_r\}$ be a basis of V, $A := \{v_1, v_2, \ldots, v_{r-1}\}$, $B \subseteq A$ and $k := |B|$. W.l.o.g. we can assume $B = \{v_1, v_2, \ldots, v_k\}$. We define a linear hyperplane E of V with $E \cap A = B$ by

$$E := \langle v_1, v_2, \ldots, v_k, v_{k+1} - v_r, v_{k+2} - v_r, \ldots, v_{r-1} - v_r \rangle.$$

One can easily check that the $r - 1$ vectors generating E are linearly independent. Thus $E \in \mathcal{E}$. Therefore it remains to show $v_{k+1}, \ldots, v_{r-1} \notin E$. Let $j \in \{k + 1, k + 2, \ldots, r - 1\}$. Assume that $v_j \in E$. Then there are $\lambda_i \in \mathbb{F}_q$, $i \in [r - 1]$ with

$$v_j = \sum_{i=1}^{k} \lambda_i v_i + \sum_{i=k+1}^{r-1} \lambda_i (v_i - v_r),$$

therefore

$$0 = \sum_{\substack{i=1 \\ i \neq j}}^{r-1} \lambda_i v_i + (\lambda_j - 1) v_j - \left(\sum_{i=k+1}^{r-1} \lambda_i \right) v_r.$$

With the linear independency of $v_1 v_2, \ldots, v_r$ we get $\lambda_j = 1$, $\lambda_i = 0$ for all $i \in [r - 1] \setminus \{j\}$ and

$$\sum_{i=k+1}^{r-1} \lambda_i = 0 \neq 1$$

which is a contradiction. Hence $E \cap A = B$ and we are done. $\square$

After determing the VC-dimension we are now able to give an upper bound for the c–color discrepancy of $\mathcal{H}$. Recall that $z = \frac{(q-1) \mod c}{c}$.

Lemma 2.6.

$$\operatorname{disc}(\mathcal{H}, c) = O\left(\left(\frac{nz}{q}\right)^{\frac{1}{2}-\frac{1}{2(r-1)}}\right).$$

Proof. In the case $z = 0$ there is nothing left to prove. Thus we can assume $z \neq 0$, i.e. $(q-1) \mod c \neq 0$. Let W be an arbitrary one-dimensional subspace of V. It is obvious that for every $E \in \mathcal{E}$ either $W \subseteq E$ or $W \cap E = \{0\}$. Thus for any two non-trivial $x, y \in W$ there exists no $E \in \mathcal{E}$ with $x \in E$ and $y \notin E$. Therefore we can give colors to all but $(q-1) \mod c$ non-trivial elements of W in such a way that every color is useded in the same amount. We do the same for every one-dimensional subspace of V. One can check that the sub-hypergraph induced by the precolored vertices has discrepancy 0. Let $\mathcal{H}' = (V', \mathcal{E}')$ be the sub-hypergraph induced by the other $n' := \frac{n-1}{q-1}x + 1$ vertices of V. Because every one-dimensional subspace of V contains at least one non-trivial element that is not precolored it is easy to see that $\mathcal{H}'$ has VC-dimension $dim(\mathcal{H}') = dim(\mathcal{H}) = r-1$. With Lemma 5.9 from [Mat99] we get

$$\pi_{\mathcal{E}'}(k) = O\left(k^{r-1}\right),$$

for all $k \in [n']$, where $\pi_{\mathcal{E}'}$ is the primal shatter function of the set system $\mathcal{E}'$ defined in the beginning of chapter 5 of [Mat99]. Now applying Theorem 3.19 from [DS2003], which is a c–color extension of Theorem 5.3 from [Mat99], we have

$$\operatorname{disc}(\mathcal{H}, c) = O\left(\left(\frac{n'}{c}\right)^{\frac{1}{2}-\frac{1}{2(r-1)}}\right) = O\left(\left(\frac{nz}{q}\right)^{\frac{1}{2}-\frac{1}{2(r-1)}}\right).$$

□

Proof of the upper bound in Theorem 2.2. Using Lemma 2.6, it suffices to prove $\left(\frac{nz}{q}\right)^{\frac{1}{2}-\frac{1}{2(r-1)}} \leq \sqrt{nz(1-z)}\left(\frac{c}{n}\right)^{\frac{1}{2(r-1)}}$. First we show $\frac{1}{q} \leq (1-z)$. If $c < q$ then $(1-z) = \frac{c-x}{c} \geq \frac{1}{c} > \frac{1}{q}$ is obvious. Thus we can assume $c \geq q$. We have $x = (q-1) \mod c = q-1$, hence $x = q-1 \leq \frac{q-1}{q}c$. Therefore

$$(1-z) = \frac{c-x}{c} \geq \frac{1}{q}.$$

Now the upper bound in Theorem 2.2 follows from

$$\begin{aligned}\left(\frac{nz}{q}\right)^{\frac{1}{2}-\frac{1}{2(r-1)}} &\leq (nz(1-z))^{\frac{1}{2}-\frac{1}{2(r-1)}} \\ &\leq \sqrt{nz(1-z)}\left(\frac{c}{n}\right)^{\frac{1}{2(r-1)}}.\end{aligned}$$

□

3 One-Sided Discrepancy of $\mathcal{H}$

Obviously the upper bound in Theorem 2.2 for the c–color discrepancy is also an upper bound for the one-sided c–color discrepancy $\mathrm{disc}^+(\mathcal{H}, c)$. But the lower bound needs to be investigated separately as we have mentioned at the end of section 1. The main theorem is:

Theorem 3.1. *Let* $z := \frac{(q-1) \bmod c}{c}$, $z \neq 0$. *If* $r \geq r_0(q) = \begin{cases} 6 & : \quad q = 2, \\ 5 & : \quad q = 3, 4, 5, \\ 4 & : \quad \textit{otherwise}, \end{cases}$

Then we have $\mathrm{disc}^+(\mathcal{H}, c) \geq \frac{\sqrt{z(1-z)}}{4q(q-1)} \sqrt{\frac{n}{c}} - 1$.

For the proof of Theorem 3.1 we need some lemmas for the Fourier transform on $\mathbb{F}_q^r$. For basics for the Fourier transform in finite groups we follow [LN94].

3.1 The Fourier Transform and Facts about $\mathbb{F}_q^r$

Let $f : V \to \mathbb{C}$ be a function. To define the Fourier Transform of f, we need the absolute tracefunction

$$\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p} : \mathbb{F}_q \to \mathbb{F}_p, \alpha \mapsto \alpha + \alpha^p + \alpha^{p^2} + \ldots + \alpha^{p^{k-1}}.$$

If it is clear which p and q are considered, we just write Tr(.). Theorem 2.23 (iii) in [LN94] states that the function $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$ is linear and onto. Now the Fourier Transform $\hat{f}$ is defined by

$$\hat{f} : V \to \mathbb{C}, z \mapsto \sum_{x \in V} f(x) e^{\frac{2\pi i}{p} \mathrm{Tr}(\langle x, z \rangle)},$$

where $\langle x, z \rangle = x_1 z_1 + x_2 z_2 + \ldots + x_r z_r$ is the common inner product in $\mathbb{F}_q^r$. Furthermore we set for all $W \leq V$:

$$\hat{f}(W) := \sum_{z \in W^\sharp} \hat{f}(z),$$

where $W^\sharp = W \setminus \{0\}$. For convenience let $\hat{A} := \hat{\mathbf{1}}_A$ be the Fourier transform of the indicator function $\mathbf{1}_A$ ($A \subseteq V$). For all $E \in \mathcal{E}$ there is a unique subspace $E^\perp$ of V of dimension one which is orthogonal to E. Lemma 3.4 together with the following definition shows the benefit of the Fourier transform for the one-sided discrepancy.

Definition 3.2. *For every* $A \subseteq V$ *and every hyperedge* $E \in \mathcal{E}$ *we define*

$$d^+(A, E) \quad := \quad |A \cap E| - \frac{|A|}{q}.$$

We need the following orthogonality relation.

Lemma 3.3. *For each* $z \in V^\sharp$ *and* $a \in V$ *we have*

$$\frac{1}{q} \sum_{x \in \langle z \rangle} e^{\frac{2\pi i}{p} \mathrm{Tr}(\langle a, x \rangle)} = \begin{cases} 0 & : \quad \langle a, z \rangle \neq 0, \\ 1 & : \quad \langle a, z \rangle = 0. \end{cases}$$

Proof. If $\langle a,z\rangle = 0$, then $\langle a,x\rangle = 0$ for every $x \in \langle z\rangle$, hence

$$\begin{aligned}\frac{1}{q}\sum_{x\in\langle z\rangle} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle a,x\rangle)} &= \frac{1}{q}\sum_{x\in\langle z\rangle} e^{\frac{2\pi i}{p}\mathrm{Tr}(0)} \\ &= \frac{1}{q}\sum_{x\in\langle z\rangle} 1 \\ &= \frac{1}{q}|\langle z\rangle| = 1.\end{aligned}$$

Now we assume $\langle a,z\rangle \neq 0$. Then $\langle a,x\rangle$ passes through the whole field $\mathbb{F}_q$, if x passes through the whole subspace $\langle z\rangle$. Because the trace function Tr a onto non-trivial maps, there exists some $y \in \langle z\rangle$ with $e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle a,y\rangle)} \neq 1$. Since the function $f : \langle z\rangle \to \langle z\rangle, x \mapsto x+y$ is bijective

$$\begin{aligned}e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle a,y\rangle)}\left(\frac{1}{q}\sum_{x\in\langle z\rangle} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle a,x\rangle)}\right) &= \frac{1}{q}\sum_{x\in\langle z\rangle} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle a,x+y\rangle)} \\ &= \frac{1}{q}\sum_{x\in\langle z\rangle} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle a,x\rangle)}.\end{aligned}$$

Thus with $e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle a,y\rangle)} \neq 1$ we get $\frac{1}{q}\sum\limits_{x\in\langle z\rangle} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle a,x\rangle)} = 0$. □

With Lemma 3.3 we establish the first link between one-sided discrepancy and Fourier analysis.

Lemma 3.4. *For every subset $A \subseteq V$ and every hyperedge $E \in \mathcal{E}$*

$$d^+(A,E) = \frac{1}{q}\hat{A}(E^\perp).$$

Proof. Let $z \in E^\perp \setminus \{0\}$. Using Lemma 3.3 we have

$$\begin{aligned}|A\cap E| &= \sum_{a\in A}\delta_{\langle a,z\rangle,0} \\ &= \sum_{a\in A}\frac{1}{q}\sum_{x\in\langle z\rangle} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle a,x\rangle)} \\ &= \frac{1}{q}\sum_{x\in E^\perp}\sum_{a\in A} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle a,x\rangle)} \\ &= \frac{1}{q}\sum_{x\in E^\perp}\hat{A}(x) \\ &= \frac{1}{q}\hat{A}(E^\perp) + \frac{1}{q}\hat{A}(0) \\ &= \frac{1}{q}\hat{A}(E^\perp) + \frac{1}{q}|A|.\end{aligned}$$

Thus

$$d^+(A,E) = |A \cap E| - \frac{1}{q}|A| = \frac{1}{q}\hat{A}(E^\perp).$$

$\square$

Note that Lemma 3.4 immediately implies the statement 6 in the introduction. We fix two parameters. Let $\alpha := |\mathcal{E}|$ and $\beta := |\{E \in \mathcal{E} \mid v \in E\}|$ for a $v \in V^\sharp = V \setminus \{0\}$.

Lemma 3.5. *We have*

(i) $\alpha = \frac{q^r-1}{q-1}$,

(ii) $\beta = \frac{q^{r-1}-1}{q-1}$.

Proof. (i) Due to orthogonality, there is a one-to-one correspondence between the subspaces of dimension one and codimension one in V. So the number of linear hyperplanes in V is the number of basis of one-dimensional subspace of V divided by the number of basis of a fixed one-dimensional subspace of V. Thus we have $\alpha = \frac{q^r-1}{q-1}$.

(ii) Double-counting gives

$$\begin{aligned}
\beta(q^r-1)+\alpha &= \left(\sum_{v\in V^\sharp} |\{E \in \mathcal{E} \mid v \in E\}|\right) + |\{E \in \mathcal{E} \mid 0 \in E\}| \\
&= \sum_{v \in V} |\{E \in \mathcal{E} \mid v \in E\}| \\
&= \sum_{E \in \mathcal{E}} |E| = \alpha q^{r-1}.
\end{aligned}$$

Thus we have $\beta = \alpha\frac{q^{r-1}-1}{q^r-1} = \frac{q^{r-1}-1}{q-1}$.

$\square$

The three statements in the next lemma will be usefull for our calculations. For convenience we define $M_2 := \left\{(a_1,a_2,k_1,k_2) \in A \times A \times \mathbb{F}_q^\sharp \times \mathbb{F}_q^\sharp \mid k_1a_1 + k_2a_2 = 0\right\}$ and $M_3 := \left\{(a_1,a_2,a_3,k_1,k_2,k_3) \in A^3 \times \left(\mathbb{F}_q^\sharp\right)^3 \mid k_1a_1+k_2a_2+k_3a_3 = 0\right\}$. Recall that $\mathcal{E}_1$ denotes the set of all one-dimensional subspaces of V.

Lemma 3.6. *Let $A \subseteq V$. It holds*

(i) $\sum_{E\in\mathcal{E}} \hat{A}(E^\perp) = n\,\mathbf{1}_A(0) - |A|$.

(ii) $\sum_{E\in\mathcal{E}} \hat{A}(E^\perp)^2 = \frac{n}{q-1}\,|M_2| - (q-1)|A|^2$.

(iii) $\sum_{E\in\mathcal{E}} \hat{A}(E^\perp)^3 = \frac{n}{q-1}\,|M_3| - (q-1)^2|A|^3$.

Proof. (i) Every vector space is the disjoint union of all its one-dimensional subspaces without the origin and the origin itself. Thus we have

$$\begin{aligned}
\sum_{E\in\mathcal{E}} \hat{A}(E^{\perp}) &= \sum_{x\in V} \hat{A}(x) \\
&= \sum_{x\in V^{\perp}} \sum_{a\in A} e^{\frac{2\pi i}{q}\mathrm{Tr}(\langle a,x\rangle)} \\
&= \sum_{a\in A} \sum_{x\in V} e^{\frac{2\pi i}{q}\mathrm{Tr}(\langle a,x\rangle)} - |A| \\
&= n\,\mathbf{1}_A(0) - |A|
\end{aligned}$$

(ii) For every $E\in\mathcal{E}$ we fix a non-trivial element $v_{E^{\perp}}\in E^{\perp}$ of the one-dimensional subspace $E^{\perp}$. Then

$$\begin{aligned}
&\sum_{E\in\mathcal{E}} \hat{A}(E^{\perp})^2 = \sum_{E\in\mathcal{E}} \left(\sum_{a\in A} \sum_{x\in E^{\perp}\setminus\{0\}} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle a,x\rangle)} \right)^2 \\
&= \sum_{E\in\mathcal{E}} \left(\sum_{a_1\in A} \sum_{x_1\in E^{\perp}\setminus\{0\}} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle a_1,x_1\rangle)} \right) \left(\sum_{a_2\in A} \sum_{x_2\in E^{\perp}\setminus\{0\}} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle a_2,x_2\rangle)} \right) \\
&= \sum_{E\in\mathcal{E}} \sum_{a_1,a_2\in A} \sum_{x_1,x_2\in E^{\perp}\setminus\{0\}} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle a_1,x_1\rangle+\langle a_2,x_2\rangle)} \\
&= \sum_{E\in\mathcal{E}} \sum_{a_1,a_2\in A} \sum_{k_1,k_2\in\mathbb{F}_q^{\sharp}} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle a_1,k_1 v_{E^{\perp}}\rangle+\langle a_2,v_{E^{\perp}}\rangle)} \\
&= \sum_{E\in\mathcal{E}} \sum_{a_1,a_2\in A} \sum_{k_1,k_2\in\mathbb{F}_q^{\sharp}} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle k_1 a_1+k_2 a_2,v_{E^{\perp}}\rangle)} \\
&= \frac{1}{q-1} \sum_{E\in\mathcal{E}} \sum_{v\in(E^{\perp})^{\sharp}} \sum_{a_1,a_2\in A} \sum_{k_1,k_2\in\mathbb{F}_q^{\sharp}} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle k_1 a_1+k_2 a_2,v\rangle)} \\
&= \frac{1}{q-1} \sum_{v\in V^{\sharp}} \sum_{a_1,a_2\in A} \sum_{k_1,k_2\in\mathbb{F}_q^{\sharp}} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle k_1 a_1+k_2 a_2,v\rangle)} \\
&= \frac{1}{q-1} \sum_{v\in V} \sum_{a_1,a_2\in A} \sum_{k_1,k_2\in\mathbb{F}_q^{\sharp}} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle k_1 a_1+k_2 a_2,v\rangle)} - (q-1)|A|^2 \\
&= \frac{n}{q-1}\,|M_2| - (q-1)|A|^2
\end{aligned}$$

(iii) For every one-dimensional subspace $W \in \mathcal{E}_1$ we fix an element $x_W \in W^\sharp$. Then

$$\begin{aligned}
\sum_{E\in\mathcal{E}} \hat{A}(E^\perp)^3 &= \sum_{E\in\mathcal{E}} \left(\sum_{x\in E^\perp\setminus\{0\}} \sum_{a\in A} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle a,x\rangle)} \right)^3 \\
&= \sum_{W\in\mathcal{E}_1} \sum_{a_1,a_2,a_3\in A} \sum_{x_1,x_2,x_3\in W^\sharp} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle a_1,x_1\rangle+\langle a_2,x_2\rangle+\langle a_3,x_3\rangle)} \\
&= \sum_{W\in\mathcal{E}_1} \sum_{a_1,a_2,a_3\in A} \sum_{k_1,k_2,k_3\in\mathbb{F}_q^\sharp} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle k_1a_1+k_2a_2+k_3a_3,x_W\rangle)} \\
&= \frac{1}{q-1} \sum_{v\in V^\sharp} \sum_{a_1,a_2,a_3\in A} \sum_{k_1,k_2,k_3\in\mathbb{F}_q^\sharp} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle k_1a_1+k_2a_2+k_3a_3,v\rangle)} \\
&= \frac{1}{q-1} \sum_{v\in V} \sum_{a_1,a_2,a_3\in A} \sum_{k_1,k_2,k_3\in\mathbb{F}_q^\sharp} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle k_1a_1+k_2a_2+k_3a_3,v\rangle)} - (q-1)^2|A|^3 \\
&= \frac{n}{q-1}|M_3| - (q-1)^2|A|^3
\end{aligned}$$

□

Calculating $|M_2|$ gives a Parseval type equation.

Corollary 3.7. *For $A \subseteq V$*

$$\sum_{E\in\mathcal{E}} |\hat{A}(E^\perp)|^2 = n(q-1)\mathbf{1}_A(0) - (q-1)|A|^2 + n \sum_{W\in\mathcal{E}_1} |A\cap W^\sharp|^2$$

and in particular

$$\sum_{E\in\mathcal{E}} |\hat{A}(E^\perp)|^2 \geq |A|(n-(q-1)|A|).$$

We proceed to the discussion of the one-sided discrepancy and show first lower bounds.

Lemma 3.8. *Let $A \subseteq V$. Then there exists an $E \in \mathcal{E}$ with*

$$d^+(A,E) \geq -1.$$

Proof. Using Lemma 3.4 we have

$$\begin{aligned}
\sum_{E\in\mathcal{E}} d^+(A,E) &= \frac{1}{q}\sum_{E\in\mathcal{E}} \hat{A}(E^\perp) \\
&= \frac{1}{q}\sum_{E\in\mathcal{E}}\sum_{z\in(E^\perp)^\sharp}\sum_{a\in A} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle a,z\rangle)} \\
&= \frac{1}{q}\sum_{a\in A}\sum_{z\in V^\sharp}\sum_{\substack{E\in\mathcal{E}\\ z\in E^\perp}} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle a,z\rangle)} \\
&= \frac{1}{q}\left(\sum_{a\in A^\sharp}\sum_{z\in V^\sharp} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle a,z\rangle)} + \mathcal{X}_A(0)\sum_{z\in V^\sharp} e^{\frac{2\pi i}{p}\mathrm{Tr}(\langle 0,z\rangle)}\right) \\
&= \frac{1}{q}\left(\sum_{a\in A^\sharp}(-1) + \mathcal{X}_A(0)(n-1)\right) \\
&= \frac{1}{q}\left(\mathcal{X}_A(0)(n-1) - |A^\sharp|\right) \\
&= \frac{1}{q}\left(\mathcal{X}_A(0)(n) - |A|\right).
\end{aligned}$$

Using $|\mathcal{E}| = \frac{n-1}{q-1}$ there is an $E\in\mathcal{E}$ with

$$d^+(A,E) \geq \frac{q-1}{n-1}\frac{1}{q}\left(\mathcal{X}_A(0)n - |A|\right).$$

For $0\in A$ we get $d^+(A,E)\geq 0$. Thus we can assume $0\notin A$. It holds

$$\begin{aligned}
d^+(A,E) &\geq -\frac{q-1}{n-1}\frac{|A|}{q} \\
&\geq -\frac{q-1}{n-1}\frac{n-1}{q} \\
&= -\frac{q-1}{q}.
\end{aligned}$$

□

Our strategy is the following: first we find a large color-class A_i. For this color-class the term $\frac{|A_i|}{q} - \frac{|E|}{c}$ is large. In the second step Lemma 3.8 gives us a hyperedge $E\in\mathcal{E}$ with $\frac{\hat{A}_i(E^\perp)}{q}\geq -1$. The next lemma will be used to find a large color-class A_i.

Lemma 3.9. *Let $k\in\mathbb{R}^+$ with $\frac{1}{q^{r-1}-q^{\frac{r}{2}-1}}\leq k\leq\frac{1}{3q}$ and $z = \frac{(q-1) \mod c}{c}$. Either there exists a color $i\in[c]$ and a hyperedge $E\in\mathcal{E}$ with*

$$|A_i\cap E| - \frac{|E|}{c} > k\frac{\sqrt{n}}{c}$$

or we have

$$\sum_{i\in[c]}\sum_{E\in\mathcal{E}}|\hat{A}_i(E^\perp)|^2 \geq \frac{n(n-1)c}{q-1}\left(z(1-z)-\tfrac{k}{c^2}(q-1)^2\right).$$

Proof. Let $A \subseteq V$. Then

$$\sum_{E\in\mathcal{E}}|A\cap E| = \sum_{a\in A}\sum_{\substack{E\in\mathcal{E}\\ a\in E}} 1 \geq \sum_{a\in A}\beta = |A|\beta.$$

Hence for every $A \subseteq V$ there exists an $E_0 \in \mathcal{E}$ with

$$|A\cap E_0| \geq |A|\frac{\beta}{\alpha} = |A|\frac{q^{r-1}-1}{q^r-1}. \tag{8}$$

Recall that $\mathcal{E}_1$ is the set of all one-dimensionalsubspaces of V. Using Corollary 3.7 we get

$$\sum_{i\in[c]}\sum_{E\in\mathcal{E}}|\hat{A}_i(E^\perp)|^2 = n(q-1) - \sum_{i\in[c]}(q-1)|A_i|^2 + n\sum_{i\in[c]}\sum_{W\in\mathcal{E}_1}|A\cap W^\sharp|^2. \tag{9}$$

We distinguish the following two cases.

Case 1: There exists an $i \in [c]$ with $|A_i| > \frac{q^r-1}{c(q^{r-1}-1)}\left(kq^{\frac{r}{2}}+q^{r-1}\right)$

Then (8) yields the existence of an $E \in \mathcal{E}$ with

$$|A_i\cap E| - \frac{|E|}{c} \geq |A_i|\frac{q^{r-1}-1}{q^r-1} - \frac{|E|}{c} \geq k\frac{\sqrt{n}}{c}.$$

Case 2: For all $i \in [c]$ we have $|A_i| \leq \frac{q^r-1}{c(q^{r-1}-1)}\left(kq^{\frac{r}{2}}+q^{r-1}\right)$

Put $x := (q-1) \mod c$ and $y := \lfloor \frac{q-1}{c} \rfloor = \frac{(q-1)-x}{c}$ and recall $n = q^r$. Using (9) we have

$$\begin{aligned}
\sum_{i\in[c]}\sum_{E\in\mathcal{E}} |\hat{A}_i(E^\perp)|^2 &= n(q-1) - (q-1)\sum_{i\in[c]} |A_i|^2 \\
&\quad + n\sum_{i\in[c]}\sum_{W\in\mathcal{E}_1} |A_i \cap W^\sharp|^2 \\
&\geq n(q-1) - (q-1)\frac{q^r-1}{c(q^{r-1}-1)}\left(kq^{\frac{r}{2}} + q^{r-1}\right)\sum_{i\in[c]} |A_i| \\
&\quad + n\sum_{W\in\mathcal{E}_1}\left((c-x)y^2 + x(y+1)^2\right) \\
&= n(q-1) - (q-1)n^2\frac{q^r-1}{c(q^{r-1}-1)}\left(\frac{k}{q^{\frac{r}{2}}} + \frac{1}{q}\right) \\
&\quad + n\frac{q^r-1}{q-1}\left(cy^2 + 2xy + x\right) \\
&\geq -n(n-1)(q-1)\frac{n-1}{c(q^{r-1}-1)}\left(\frac{k}{q^{\frac{r}{2}}} + \frac{1}{q}\right) \\
&\quad + n\frac{n-1}{q-1}\left(\frac{((q-1)-x)^2}{c} + \frac{2x((q-1)-x)}{c} + x\right) \\
&= n(n-1)\left(\frac{(q-1)^2 - x^2 + cx}{c(q-1)} - \frac{(q-1)(q^r-1)}{c(q^{r-1}-1)}\left(\frac{k}{q^{\frac{r}{2}}} + \frac{1}{q}\right)\right)
\end{aligned}$$

Now $q^{r-1} - q^{\frac{r}{2}} - 1 \geq \frac{1}{k}$ yields

$$-kq^{\frac{3}{2}r-1} + kq^r + kq^{\frac{r}{2}} + q^{\frac{r}{2}} \leq 0.$$

Using this inequality we get

$$\begin{aligned}
(q^r-1)(k+q^{\frac{r}{2}-1}) - (1+k)(q^{r-1}-1)q^{\frac{r}{2}} &= kq^r - k - q^{\frac{r}{2}-1} - kq^{\frac{3}{2}r-1} - q^{\frac{r}{2}} - kq^{\frac{r}{2}} \\
&\leq -kq^{\frac{3}{2}r-1} + kq^r + kq^{\frac{r}{2}} + q^{\frac{r}{2}} \leq 0,
\end{aligned}$$

and hence

$$\frac{q^r-1}{q^{r-1}-1}\left(\frac{k}{q^{\frac{r}{2}}} + \frac{1}{q}\right) \leq 1 + k. \tag{10}$$

Thus we have

$$\begin{aligned}
\sum_{i\in[c]}\sum_{E\in\mathcal{E}} |\hat{A}_i(E^\perp)|^2 &\geq n(n-1)\left(\frac{cx + (q-1)^2 - x^2}{c(q-1)} - \frac{(q-1)(q^r-1)}{c(q^{r-1}-1)}\left(\frac{k}{q^{\frac{r}{2}}} + \frac{1}{q}\right)\right) \\
&\overset{(10)}{\geq} n(n-1)\left(\frac{cx + (q-1)^2 - x^2}{c(q-1)} - \frac{(q-1)}{c}(1+k)\right) \\
&= n(n-1)\frac{cx - k(q-1)^2 - x^2}{c(q-1)} \\
&= \frac{n(n-1)c}{q-1}\left(z(1-z) - \tfrac{k}{c^2}(q-1^2)\right).
\end{aligned}$$

□

3.2 Proof of the Main Theorem

For the proof of Theorem 3.1 we need some notations. Put for all $E \in \mathcal{E}$

$$\begin{aligned} I^+(E) &:= \{i \in [c] \mid \hat{A}(E^\perp) \geq 0\}, \\ I^-(E) &:= [c] \setminus I^+(E), \\ M &:= \max_{i \in [c]} \max_{E \in \mathcal{E}} |\hat{A}_i(E^\perp)|. \end{aligned}$$

It is straightforward to see that there is an $E \in \mathcal{E}$ with

$$\sum_{i \in I^+(E)} \hat{A}_i(E^\perp) = -\sum_{i \in I^-(E)} \hat{A}_i(E^\perp) \geq M. \tag{11}$$

Proof of Theorem 3.1. Let $r \geq r_0(q)$ and $\chi : V \to [c]$ be a c–coloring of $\mathcal{H}$. Set $k := \frac{\sqrt{cz(1-z)}}{4q(q-1)}$. We have to show

$$\operatorname{disc}^+(\mathcal{H}, c) \geq k\frac{\sqrt{n}}{c}.$$

For convenience we fix another constant $\xi := \sqrt{\frac{nc}{8q}\left(z(1-z) - \frac{k}{c^2}(q-1)^2\right)}$. Straightforward calculation shows that the radian is positive. We want to show

$$\frac{\xi}{c} \geq k\frac{\sqrt{n}}{c}, \tag{12}$$

which we will use later on. Both numbers are positive, therefore it suffice to prove $\frac{\xi^2}{n} - k^2 \geq 0$. We have

$$\begin{aligned} \frac{\xi^2}{n} - k^2 &= \frac{c}{8q}\left(z(1-z) - \frac{\sqrt{cz(1-z)}(q-1)}{4qc^2}\right) - \frac{cz(1-z)}{16q^2(q-1)^2} \\ &= \frac{c}{8q}\left[z(1-z) - \frac{\sqrt{cz(1-z)}(q-1)}{4qc^2} - \frac{z(1-z)}{2q(q-1)^2}\right] \\ &\geq \frac{c}{8q}\left[z(1-z) - \frac{z(1-z)}{4} - \frac{z(1-z)}{4}\right] \\ &= \frac{cz(1-z)}{16q} \geq 0. \end{aligned}$$

We make the following nested case distinctions.

Case 1: $\sum_{i \in I^+(E)} \left(|A_i \cap E| - \frac{|E|}{c}\right) \geq \xi$ for an $E \in \mathcal{E}$.

In this case we have

$$c \operatorname{disc}^+(\mathcal{H}, c, \chi) \geq \sum_{i \in I^+(E)} \left(|A_i \cap E| - \frac{|E|}{c}\right) \geq \xi,$$

hence

$$\operatorname{disc}^+(\mathcal{H}, c, \chi) \geq \frac{\xi}{c} \overset{(12)}{\geq} k\frac{\sqrt{n}}{c}. \tag{13}$$

Case 2: $\sum_{i\in I^+(E)}\left(|A_i\cap E|-\frac{|E|}{c}\right)<\xi$ for all $E\in\mathcal{E}$.

It follows

$$\sum_{i\in I^-(E)}\left(|A_i\cap E|-\frac{|E|}{c}\right)=-\sum_{i\in I^+(E)}\left(|A_i\cap E|-\frac{|E|}{c}\right)>-\xi \tag{14}$$

for each $E\in\mathcal{E}$.

Case 2.1: $M\geq 2q\xi$.

By (6) and (11) we have for an appropriate $E\in\mathcal{E}$:

$$\begin{aligned}\frac{1}{q}\sum_{i\in I^-(E)}|A_i|-|I^-(E)|\frac{|E|}{c} &= \sum_{i\in I^-(E)}\left(|A_i\cap E|-\frac{|E|}{c}\right)-\frac{1}{q}\sum_{i\in I^-(E)}\hat{A}_i(E^\perp)\\ &\overset{(14)}{>} -\xi+\frac{1}{q}M\\ &\geq -\xi+\frac{2q\xi}{q}\\ &= \xi.\end{aligned}$$

Hence there exists an $i_0\in I^-(X)$ with

$$\frac{1}{q}|A_{i_0}|-\frac{|E|}{c}\geq\frac{\xi}{c}\overset{(12)}{\geq}k\frac{\sqrt{n}}{c}.$$

Lemma 3.8 ensures the existence of an $E_0\in\mathcal{E}$ with

$$\frac{1}{q}\hat{A}_{i_0}(E_0^\perp)\geq -1,$$

thus

$$\begin{aligned}\mathrm{disc}^+(\mathcal{H},c,\chi)\geq|A_{i_0}\cap E_0|-\frac{|E_0|}{c} &= \frac{1}{q}\hat{A}_{i_0}(E_0^\perp)+\frac{1}{q}|A_{i_0}|-\frac{|E_0|}{c}\\ &\geq k\frac{\sqrt{n}}{c}-1.\end{aligned} \tag{15}$$

Case 2.2: $M<2q\xi$.

To use Lemma 3.9 we have to verify $\frac{1}{q^{r-1}-q^{\frac{r}{2}-1}}\leq k\leq\frac{1}{3q}$. It holds $cz\leq q-1$ and $1-z\leq 1$. Therfore we get

$$k=\frac{\sqrt{cz(1-z)}}{4q(q-1)}\leq\frac{\sqrt{q-1}}{4q(q-1)}<\frac{1}{3q}.$$

Using $z\neq 0$ we get a lower bound for k:

$$k\geq\frac{\sqrt{\frac{c-1}{c}}}{4q(q-1)}\geq\frac{1}{4\sqrt{2}q(q-1)}>\frac{1}{6q(q-1)}.$$

The only thing that is left to prove is $\frac{1}{6q(q-1)} \geq \frac{1}{q^{r-1}-q^{\frac{r}{2}}-1}$. But it is easy to check that $r \geq r_0(q)$ assures this inequality. Thus we can apply Lemma 3.9. Hence either there exists a color $i \in [c]$ and an $E \in \mathcal{E}$ such that

$$|A_i \cap E| - \frac{|E|}{c} > k\frac{\sqrt{n}}{c} \tag{16}$$

or we have

$$\sum_{i\in[c]}\sum_{E\in\mathcal{E}} |\hat{A}_i(E^\perp)|^2 \geq \frac{n(n-1)c}{q-1}\left(z(1-z) - \frac{k}{c^2}(q-1)^2\right). \tag{17}$$

In the first case we get a lower bound for the one-sided discrepancy as desired. Therefore we can assume that (2) holds. Then there exists an $E \in \mathcal{E}$ with

$$\sum_{i\in[c]} |\hat{A}_i(E^\perp)|^2 \geq nc\left(z(1-z) - \frac{k}{c^2}(q-1)^2\right).$$

Hence we get

$$\begin{aligned}
2M\left(\sum_{i\in I^-(E)} |\hat{A}^i(E^\perp)|\right) &= M\left(\sum_{i\in I^-(E)} |\hat{A}^i(E^\perp)| + \sum_{i\in I^+(E)} |\hat{A}^i(E^\perp)|\right) \\
&= M\sum_{i\in[c]} |\hat{A}^i(E^\perp)| \\
&\geq \sum_{i\in[c]} |\hat{A}^i(E^\perp)|^2 \\
&\geq nc\left(z(1-z) - \frac{k}{c^2}(q-1)^2\right).
\end{aligned}$$

Thus we get

$$\sum_{i\in I^-(E)} |\hat{A}^i(E^\perp)| \geq \frac{nc}{2M}\left(z(1-z) - \frac{k}{c^2}(q-1)^2\right) > \frac{nc}{4q\xi}\left(z(1-z) - \frac{k}{c^2}(q-1)^2\right) = 2\xi$$

and (6) yields

$$\begin{aligned}
\frac{1}{q}\sum_{i\in I^-(E)} |A_i| - |I^-(E)|\frac{|E|}{c} &= \sum_{i\in I^-(E)}\left(|A_i \cap E| - \frac{|E|}{c}\right) - \frac{1}{q}\sum_{i\in I^-(E)} \hat{A}_i(E^\perp) \\
&\overset{(14)}{>} -\xi + 2\xi = \xi.
\end{aligned}$$

Hence there exists an $i_0 \in I^-(X)$ such that

$$\frac{1}{q}|A_{i_0}| - \frac{|E|}{c} > \frac{\xi}{c} \overset{(12)}{\geq} k\frac{\sqrt{n}}{c}.$$

Lemma 3.8 provides the existence of an $E_0 \in \mathcal{E}$ with

$$\frac{1}{q}\hat{A}_{i_0}(E_0^{\perp}) \geq -1,$$

and we have

$$\begin{aligned} \mathrm{disc}^+(\mathcal{H}, c, \chi) &\geq |A_{i_0} \cap E_0| - \frac{|E_0|}{c} \\ &= \frac{1}{q}\hat{A}_{i_0}(E_0^{\perp}) + \frac{1}{q}|A_{i_0}| - \frac{|E|}{c} \\ &\geq k\frac{\sqrt{n}}{c} - 1. \end{aligned} \tag{18}$$

We have shown

$$\mathrm{disc}^+(\mathcal{H}, c, \chi) \geq k\frac{\sqrt{n}}{c} - 1 = \frac{\sqrt{z(1-z)}}{4q(q-1)} - 1.$$

□

4 One-Sided Discrepancy for a Large Number of Colors

One factor in the lower bound of the one-sided discrepancy is the term $\sqrt{z(1-z)}$ with $z = \frac{(q-1) \mod c}{c}$. Therefore we get no lower bound if $c|(q-1)$ as we have already mentioned. Moreover, if $(q-1) \mod c$ is either small (almost zero) or large (almost c), the lower bound is of order $\Omega(\frac{\sqrt{n}}{c})$. This is a factor of $\sqrt{c}$ smaller than the upper bound $O(\sqrt{\frac{n}{c}})$. This problem arises for instance if the number of colors c is greater or equal than q. In this situation we have $\sqrt{z(1-z)} \leq \sqrt{\frac{q-1}{c}}$ and the lower bound in Theorem 3.1 becomes less or equal than

$$\frac{1}{4q\sqrt{q-1}}\frac{\sqrt{n}}{c} - 1.$$

Thus it is only of order $\Omega(\frac{\sqrt{n}}{c})$. But if the number of colors is large enough, presisely if $c \geq qn^{\frac{1}{3}}$, we can close this gap.

Theorem 4.1. *Let $c \geq qn^{1/3}$ and $r \geq 4$. Then*

$$\mathrm{disc}^+(\mathcal{H}, c) \geq \frac{1}{22\sqrt{q}}\sqrt{\frac{n}{c}} - 1.$$

In particular

$$\mathrm{disc}^+(\mathcal{H}, c) = \Theta(\sqrt{\tfrac{n}{c}}).$$

The key for the proof of Theorem 4.1 is the following lemma.

Lemma 4.2. *Let $A \subseteq V$ with $|A| \leq \frac{1}{2}q^{r-1}$ and $0 \in A$. There exists an $E \in \mathcal{E}$ with*

$$d^+(A,E) \geq \min\left\{\frac{1}{16(q-1)^2}\frac{n}{|A|}, \frac{1}{3\sqrt{q}}\sqrt{|A|}\right\}.$$

Proof. Corollary 3.7 yields

$$\sum_{E\in\mathcal{E}} |\hat{A}(E^\perp)|^2 \geq |A|(n-(q-1)|A|) \geq \tfrac{1}{2}n|A|.$$

Let us denote by $\mathcal{E}^+$ the set of all $E \in \mathcal{E}$ with $\hat{A}(E^\perp) \geq 0$ and by $\mathcal{E}^-$ the set of all $E \in \mathcal{E}$ with $\hat{A}(E^\perp) < 0$. Furthermore define $M := \max_{E\in\mathcal{E}} \hat{A}(E^\perp)$. Recall that we have defined $\alpha = |\mathcal{E}| = \frac{n-1}{q-1} \leq 2q^{r-1}$. Lemma 3.6 (i) gives

$$\begin{aligned}
\sum_{E\in\mathcal{E}^-} \left|\hat{A}(E^\perp)\right| &= -\sum_{E\in\mathcal{E}} \hat{A}(E^\perp) + \sum_{E\in\mathcal{E}^+} \hat{A}(E^\perp) \\
&= |A| - n\chi_A(0) + \sum_{E\in\mathcal{E}^+} \hat{A}(E^\perp) \\
&\leq \alpha M.
\end{aligned}$$

Using Lemma 3.6 (iii) in the same way, we get

$$\sum_{E\in\mathcal{E}^-} \left|\hat{A}(E^\perp)\right|^3 = \alpha M^3 + (q-1)^2|A|^2$$

Thus, the Cauchy-Schwarz inequality gives

$$\begin{aligned}
\tfrac{1}{2}n|A| &\leq \sum_{E\in\mathcal{E}^+} |\hat{A}(E^\perp)|^2 + \sum_{E\in\mathcal{E}^-} |\hat{A}(E^\perp)|^2 \\
&\leq \alpha M^2 + \sum_{E\in\mathcal{E}^-} |\hat{A}(E^\perp)|^{\frac{1}{2}} |\hat{A}(E^\perp)|^{\frac{3}{2}} \\
&\leq \alpha M^2 + \left(\sum_{E\in\mathcal{E}^-} |\hat{A}(E^\perp)|\right)^{\frac{1}{2}} \left(\sum_{E\in\mathcal{E}^-} |\hat{A}(E^\perp)|^3\right)^{\frac{1}{2}} \\
&\leq \alpha M^2 + (\alpha M)^{\frac{1}{2}} \left((q-1)^2|A|^3 + \alpha M^3\right)^{\frac{1}{2}}.
\end{aligned}$$

Case 1: $(q-1)^2|A|^3 < \alpha M^3$

Here get

$$\begin{aligned}
\frac{1}{2}n|A| &\leq \alpha M^2 + (\alpha M)^{\frac{1}{2}}(2\alpha M^3)^{\frac{1}{2}} \\
&\leq (1+\sqrt{2})\alpha M^2,
\end{aligned}$$

and thus $M \geq \sqrt{\frac{q|A|}{10}}$.

Case 2: $(q-1)^2|A|^3 \geq \alpha M^3$

We distinguish two subcases. Let us first assume that $\alpha M^2 \geq (\alpha M)^{\frac{1}{2}} \left(2(q-1)^2|A|^3\right)^{\frac{1}{2}}$. Then we have

$$M \geq \sqrt{\frac{n|A|}{4\alpha}} \geq \sqrt{\frac{q|A|}{8}}.$$

The other case is $\alpha M^2 < (\alpha M)^{\frac{1}{2}} \left(2(q-1)^2|A|^3\right)^{\frac{1}{2}}$. Here we have

$$\frac{1}{2}n|A| \quad \leq \quad \left(\alpha M 2(q-1)^2|A|^3\right)^{\frac{1}{2}},$$

and alltogether

$$M \geq \frac{q^{2r}}{|A|8\alpha(q-1)^2} \geq \frac{q}{16(q-1)^2}\frac{n}{|A|}.$$

□

Now we are able to prove Theorem 4.1.

Proof of Theorem 4.1. Let $\chi : V \to [c]$ be a c–coloring of $\mathcal{H}$. Let $A_i := \chi^{-1}(i)$ for all $i \in [c]$. There exists at least one color-class A_i with $|A_i| \geq \frac{n}{c}$. In the case that there is no color-class A_i with $\frac{n}{c} \leq |A_i \cup \{0\}| \leq \frac{n}{c} + \frac{1}{3}\sqrt{\frac{n}{c}}$ there must be a color-class A_{i_0} with $|A_{i_0} \cup \{0\}| > \frac{n}{c} + \frac{1}{3}\sqrt{\frac{n}{c}}$. Using Lemma 3.8 we get

$$\mathrm{disc}^+(\mathcal{H}, c, \chi) \geq \max_{E \in \mathcal{E}} \left(d^+\left(A_{i_0} \cup \{0\}, E\right) + \frac{\left|A_{i_0} \cup \{0\}\right|}{q} - \frac{|E|}{c} \right) - 1 \geq \frac{1}{3q}\sqrt{\frac{n}{c}} - 1.$$

Therefore we can assume the existence of a color-class A_{i_0} with $\frac{n}{c} \leq |A_{i_0} \cup \{0\}| \leq \frac{n}{c} + \frac{1}{3}\sqrt{\frac{n}{c}}$. To apply Lemma 4.2 we have to assure $|A_{i_0} \cup \{0\}| < \frac{1}{2}q^{r-1}$. We get

$$|A_{i_0} \cup \{0\}| \quad \leq \quad \frac{n}{c} + \frac{1}{3}\sqrt{\frac{n}{c}} \leq q^{r-\frac{4}{3}-1} + \frac{1}{3}q^{\frac{r}{2}-\frac{2}{3}-\frac{1}{2}} \leq q^{r-1}\left(2^{-\frac{4}{3}} + \frac{1}{3}2^{-\frac{13}{6}}\right) < \frac{1}{2}q^{r-1}.$$

Hence there exists an $E \in \mathcal{E}$ with

$$d^+(A_{i_0} \cup \{0\}, E) \geq \min\left\{ \frac{1}{16(q-1)^2}\frac{n}{|A_{i_0} \cup \{0\}|}, \frac{1}{3\sqrt{q}}\sqrt{|A_{i_0} \cup \{0\}|} \right\} \geq \frac{1}{22\sqrt{q}}\sqrt{\frac{n}{c}}.$$

Thus, we have

$$\mathrm{disc}^+(\mathcal{H}, c, \chi)) \geq d^+(A_{i_0} \cup \{0\}, E) + \frac{|A_{i_0}|}{q} - \frac{|E|}{c} - 1 \geq \frac{1}{22\sqrt{q}}\sqrt{\frac{n}{c}} - 1.$$

□

5 Conclusion

We have shown that the c-color discrepancy of the hypergraph of linear hyperplanes in $\mathbb{F}_q^r$ is tight. For the one-sided c-color discrepancy of the same hypergraph we have given a lower and an upper bound which differ by a factor a $\sqrt{c}$. We have closed this gap in the case that the number of colors c is large ($c \geq qn^{\frac{1}{3}}$). The challenging algorithmic problem still is to construct an $O(\sqrt{n})$–discrepancy coloring for the linear hyperplanes in $\mathbb{F}_q^r$. Furthermore, a more detailed study of one-sided discrepancy is desirable.

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