A Cauchy problem for the Helmholtz equation: application to analysis of light propagation in solids

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Abstract

A problem of determination of radiation field inside a solid from experimental data given on a part of surface surrounding this solid is considered. The model problem has been formulated as a Cauchy problem for the Helmholtz equation. For solving it, an approximate method based on regularization in frequency space is formulated and analyzed. Convergence and stability of the method are proved under a suitable choice of regularization parameter and numerical implementation of the method is presented. Possible application of the method to problems of propagation of laser beams in solids is discussed.

Key words: Light beam propagation, laser beam, ill-posed problem, Cauchy problem for elliptic equation, Helmholtz equation, regularization method

1 Introduction

Precise determination of radiation field inside a solid is a difficult task, because, as a rule, we are not able to measure such an electromagnetic field directly. Practically, we are able to measure the field only on certain subsets of physical

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space (e.g. on some surfaces) surrounding the solid. Therefore, the problem arises how to reconstruct the radiation field from such experimental data.(see for instance [1], [17]).

In this paper we consider a model problem that corresponds to a frequently occurring physical situation: A light beam (e.g. a laser beam) penetrates a solid from the left and emerges at the surface Γ_S and then propagates in the air (see Fig. 1). We assume hat the solid is homogenous and isotropic with the dielectric constant $\varepsilon = \varepsilon_1$. (Magnetic susceptibility $\mu = 1$). In the air we have naturally $\varepsilon = 1$ and $\mu = 1$. We can measure certain components of the electromagnetic field on certain surfaces Γ, Γ' situated not far from the side-face of the solid. Employing the experimental data we define simultaneously the Dirichlet and the Neuman boundary conditions on the surface Γ. On the surface Γ_S we must formulate the relevant continuity conditions. It should be emphasized that the surfaces Γ, Γ' do not surround the domain where the solution is looked for. So, our approach differs substantially from the standard one, when the problem of light propagation is formulated in terms of the Neuman or Dirichlet boundary conditions (sometimes supplemented by the Sommerfeld radiation conditions)on certain surfaces totally surrounding the considered domain.

Let us also emphasize that the two approaches have different areas of applications . The standard approach is applicable in the case when full information about the electromagnetic field on the boundary surrounding the considered domain is known (or, at least, assumed). In such a case we can,in principle, determine the electromagnetic field in all considered domain. In contrast, in our approach, only partial information about boundary values of electromagnetic field is given, and, as a result, we can reconstruct the electromagnetic field only in certain part of physical space. It has to be mentioned that the standard approach is connected with several difficulties: As a rule, the relevant boundary conditions are not known on the whole assumed boundary. In such a case, on parts of the boundary different modifications of Sommerfeld radiation conditions are formulated. However, justification of such artificial boundary conditions needs certain strong a priori assumptions. Especially difficult task is the formulation of the relevant Sommerfeld radiation conditions for the domains which are not bounded regions ([9] , [18]). Nevertheless, the standard approach leads to the well-posed boundary value problems which enables applying the numerical analysis for solving them.

In contrast, our approach leads to the ill-posed boundary value problem, i.e. the solution of it does not depend continuously on the boundary data and small errors in these data can destroy the numerical solution (cf. [11]). For numerical solving of such a problem so-called regularization method should be applied (see [8] for a wide review of regularization methods).

Presented method enables us to find a solution only in a part of physical space. This part is shown in Fig.1 as a region $\Omega_0 \cup \Gamma_S \cup \Omega_1$. The size of this region cannot be given a priori and will be determined in Section 4. Moreover, we must accept certain assumptions concerning geometry of our physical system and, as a result, determine the relevant space of functions in which the boundary value problem will be formulated. We assume that the radius of the beam is small comparing with the transversal dimension of the solid. We assume further that the region Ω_0 is far from the sources of electromagnetic field, so we need not include any sources into our considerations. It seems to be acceptable that in case of narrow beam, the boundary conditions far from its axis do not play substantial role. So, the problem of boundary values at infinity can be reduced to the question of appropriate choosing of the class of the solution.

In our opinion, the proposed method corresponds well with the real procedure of physical measurements. In fact, we can measure the electromagnetic field only on certain parts of the surfaces surrounding the solid and the results of measurements contain some errors. Therefore, the ill-posedness is the natural consequence of the physical situation. From the point of view of physics it is also understandable that the information obtained in the described process of measurement is insufficient for determining the electromagnetic field uniquely. So, some a priori assumptions about the solution should be introduced, which is characteristic of ill-posed problems. This matter will be discussed in detail in Section 4.

The paper is organized as follows. In Section 2 we describe in detail the model physical system and formulate the model problem in terms of the Cauchy problem for the Helmholtz equation. The relevant boundary conditions and the continuity conditions on the surface of material discontinuity are given. The ill-posedness of the model problem is discussed in Sections 3 and 4: In Section 3 an auxiliary Cauchy problem for the Helmholtz equation with a constant wavenumber k is considered. The regularization method based on truncated Fourier transform is proposed in Section 4. This section contains the main theorems concerning convergence and stability of the method.

In this work only theoretical analysis of the problem is presented. Numerical calculations illustrating the obtained results will be presented it the forthcoming full version of the paper.

2 Boundary value problem

In this article, we shall consider only stationary processes (for definitions see [3]). In such a case, the vectors of electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ and of magnetic field

$\mathbf{H}(\mathbf{r}, t)$ have the form

$$\boldsymbol{E}(\boldsymbol{r}, t) = e^{-i\omega t}\boldsymbol{E}(\boldsymbol{r}), \qquad \boldsymbol{H}(\boldsymbol{r}, t) = e^{-i\omega t}\boldsymbol{H}(\boldsymbol{r}), \qquad \boldsymbol{r} = (x, y, z), \tag{1}$$

where ω is a constant frequency, and the Maxwell's equations lead to the Helmholtz equations for the vectors $\boldsymbol{E}(\boldsymbol{r})$ and $\boldsymbol{H}(\boldsymbol{r})$:

$$\Delta\boldsymbol{E}(\boldsymbol{r}) + k^2\boldsymbol{E}(\boldsymbol{r}) = 0, \;\; \Delta\boldsymbol{H}(\boldsymbol{r}) + k^2\boldsymbol{H}(\boldsymbol{r}) = 0 \qquad \text{for } \boldsymbol{r} \in \Omega \subset \mathbb{R}^3. \tag{2}$$

Here

$$k^2 = \varepsilon\mu\frac{\omega^2}{c^2}, \tag{3}$$

and the domain Ω depends on the considered problem [12].

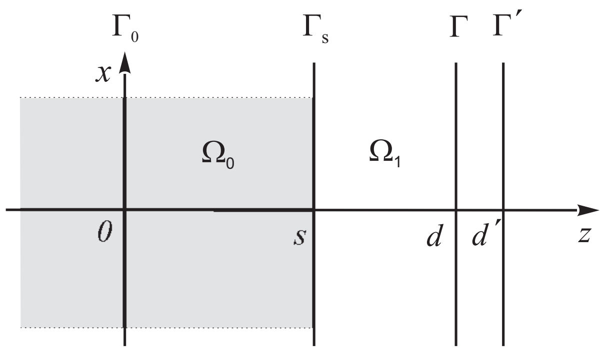

Fig. 1. Geometry of the model problem. The half-space $z < s$ is filled by a solid, the half-space $z > s$ is occupied by the air. The light beam propagates from the left to the right.

In case when boundary conditions for the fields $\boldsymbol{E}(\boldsymbol{r})$ and $\boldsymbol{H}(\boldsymbol{r})$ are linear, we can formulate boundary value problems for each component of electromagnetic field separately. Taking into account the geometry of our physical system, it is convenient to formulate the boundary value problem for the electric field component $E_x(\boldsymbol{r})$. This component is tangential to the side-face of the solid Γ_S, so continuity conditions on the surface can easily be formulated. In the sequel the electric field component $E_x(\boldsymbol{r})$ will be denoted by $u(\boldsymbol{r})$.

Let us consider the system presented in Fig.1 in more detail. In the picture Γ_0, Γ_S, Γ, $\Gamma^{'}$ are parallel surfaces in the space $\mathbb{R}^3$:

$$\begin{cases} \Gamma_0 = \{\boldsymbol{r} \in \mathbb{R}^3 : z = 0\}, \;\; \Gamma_S = \{\boldsymbol{r} \in \mathbb{R}^3 : z = s\}, \\ \Gamma = \{\boldsymbol{r} \in \mathbb{R}^3 : z = d\}, \quad \Gamma^{'} = \{\boldsymbol{r} \in \mathbb{R}^3 : z = d^{'}\}. \end{cases} \tag{4}$$

Let Ω_0 denote the part of the space $\mathbb{R}^3$ contained between the planes Γ_0 and Γ_S, and let Ω_1 be the part of the space $\mathbb{R}^3$ contained between the planes

Γ_S and Γ. (The surface Γ' is an auxiliary surface for determining $\partial_z u$ from experimental data). Let $\Omega = \Omega_0 \cup \Gamma_S \cup \Omega_1$.

Our boundary value problem can be formulated as follows:

We are looking for the function $u(\boldsymbol{r})$ satisfying the Cauchy problem for the Helmholtz equation:

$$\begin{cases} \Delta u(\boldsymbol{r}) + k^2 u(\boldsymbol{r}) = 0 & \text{in } \Omega, \\ u(\boldsymbol{r}) = g(\boldsymbol{r}) & \text{for } \boldsymbol{r} \in \Gamma, \\ \partial_z u(\boldsymbol{r}) = h(\boldsymbol{r}) & \text{for } \boldsymbol{r} \in \Gamma. \end{cases} \tag{5}$$

Here the wavenumber $k = k(\boldsymbol{r})$ is a step function:

$$k^2 = \begin{cases} k_1^2 & \text{in } \Omega_0, \\ k_2^2 & \text{in } \Omega_1. \end{cases} \tag{6}$$

The function $u(\boldsymbol{r})$ must satisfy the usual continuity conditions on the surface of material discontinuity Γ_S (see [5], [13]). Therefore, the following continuity conditions are assumed:

$$[u(\boldsymbol{r})] = [\partial u(\boldsymbol{r})/\partial \boldsymbol{n}] = 0 \qquad \text{on } \Gamma_S. \tag{7}$$

Here $\boldsymbol{n}$ is the unit normal to the surface Γ_S and $[\ .\ ]$ denote the jumps of the relevant functions when crossing this surface. (We mentioned in Introduction that we would consider only homogeneous isotropic media. Constitutive equatons for anisotropic non-homogeneous medium can be found e.g. in [14].)

3 Auxiliary problem with a constant wavenumber

In this section we will consider the Helmholtz equation with a constant wavenumber k in the domain $\Omega = \mathbb{R}^2 \times (0, d) \subset \mathbb{R}^3$, $d > 0$. For simplicity, the first two variables will be denoted by $\boldsymbol{\rho} = (x, y)$. Let $\Gamma_0 := \{(\boldsymbol{\rho}, 0), \boldsymbol{\rho} \in \mathbb{R}^2\} \subset \partial\Omega$, $\Gamma := \{(\boldsymbol{\rho}, d), \boldsymbol{\rho} \in \mathbb{R}^2\} \subset \partial\Omega$. Let $v \in H^2(\Omega)$ be a solution of the following Cauchy problem

$$\begin{cases} \Delta v + k^2 v = 0, & \text{in } \Omega \\ v(\boldsymbol{\rho}, d) = g(\boldsymbol{\rho}) & \boldsymbol{\rho} \in \mathbb{R}^2, \\ \partial_z v(\boldsymbol{\rho}, d) = 0 & \boldsymbol{\rho} \in \mathbb{R}^2, \end{cases} \tag{8}$$

where $g, \in L^2(\mathbb{R}^2)$ are given data. We assume that for the exact data the unique solution exists in $H^2(\Omega)$. We look for an approximate solution inside

Ω in the case when the data are given approximately, i.e. when $g_\delta \in L^2(\mathbb{R}^2)$ are used as the data and

$$\|g_\delta - g\|_{L^2(\mathbb{R}^2)} \leq \delta \tag{9}$$

The problem above was considered in [16]. It was shown there that the problem is ill-posed in the Hadamard sense, i.e. the solution does not depend continuously on the boundary data and small errors in the data can destroy the numerical solution. For numerical solving this problem in a stable way we have applied a method of regularization in the frequency space, which consists in cutting off high frequencies. This approach was successfully used for sideways heat equation, see for instance [7]. The literature concerning the Cauchy problem for elliptic equation is devoted mainly the Laplace equation (see for instance [2] and the references given there). For the Helmholtz equation, an influence of the frequency k on the stability of Cauchy problems was described in [10]. Moreover, other ill-posed problems for the Helmholtz equation were extensively studied in literature, among others: inverse problem of determining the shape of a part of boundary [4], inverse problem of determination of sources [15], [6].

In this Section we collect notations, definitions and results from [16], which are based for formulation and proving stability and convergence of regularization method for (5).

We make a Fourier transform with respect to $\boldsymbol{\rho} \in \mathbb{R}^2$

$$\mathcal{F}v(\boldsymbol{\xi}, z) = \widehat{v}(\boldsymbol{\xi}, z) = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} v(\boldsymbol{\rho}, z) e^{-i\boldsymbol{\xi}\cdot\boldsymbol{\rho}} d\boldsymbol{\rho},$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $\boldsymbol{\xi}\cdot\boldsymbol{\rho} = \xi_1 x + \xi_2 y$. Under the smoothness assumption

$$\mathcal{F}(\frac{\partial^2}{\partial x^2} v) = -\boldsymbol{\xi}_1^2 \mathcal{F}v;\ \mathcal{F}(\frac{\partial^2}{\partial y^2} v) = -\boldsymbol{\xi}_2^2 \mathcal{F}v;\ \mathcal{F}(\partial_z v) = \partial_z \mathcal{F}v.$$

Hence, for each $\boldsymbol{\xi} \in \mathbb{R}^2$ the function $\widehat{v}(\boldsymbol{\xi}, \cdot)$ is characterized by the following Cauchy problem

$$\begin{cases} \widehat{v}_{zz}(\boldsymbol{\xi}, z) = (|\boldsymbol{\xi}|^2 - k^2)\, \widehat{v}(\boldsymbol{\xi}, z), & \boldsymbol{\xi} \in \mathbb{R}^2,\ z \in (0, d) \\ \widehat{v}(\boldsymbol{\xi}, d) = \widehat{g}(\boldsymbol{\xi}) & \boldsymbol{\xi} \in \mathbb{R}^2, \\ \partial_z \widehat{v}(\boldsymbol{\xi}, d) = 0 & \boldsymbol{\xi} \in \mathbb{R}^2. \end{cases} \tag{10}$$

Thus,

$$\widehat{v}(\boldsymbol{\xi}, z) = \widehat{g}(\boldsymbol{\xi}) \cosh((d - z)\sqrt{|\boldsymbol{\xi}|^2 - k^2}), \tag{11}$$

$$\partial_z \widehat{v}(\boldsymbol{\xi}, z) = \widehat{g}(\boldsymbol{\xi})\sqrt{|\boldsymbol{\xi}|^2 - k^2}\sinh((d-z)\sqrt{|\boldsymbol{\xi}|^2 - k^2}). \tag{12}$$

Let us introduce the following family of discs in $\mathbb{R}^2$:

$$S_\alpha = \{\boldsymbol{\xi} \in \mathbb{R}^2 : |\boldsymbol{\xi}|^2 \leq \alpha\},$$

parameterized by the parameter $\alpha \in \mathbb{R}^+$. Let for $\delta \geq 0$

$$\widehat{g}_\alpha^\delta(\boldsymbol{\xi}) := \begin{cases} \widehat{g_\delta}(\boldsymbol{\xi}) & \text{for } \boldsymbol{\xi} \in S_\alpha, \\ 0 & \text{for } \boldsymbol{\xi} \in \mathbb{R}^2 \setminus S_\alpha \end{cases} \tag{13}$$

and $\widehat{g}_\alpha := \widehat{g}_\alpha^0$. Then $\widehat{v}_\alpha^\delta$ given by the formula

$$\widehat{v}_\alpha^\delta(\boldsymbol{\xi}, z) = \widehat{g}_\alpha^\delta(\boldsymbol{\xi})\cosh\left((d-z)\sqrt{|\boldsymbol{\xi}|^2 - k^2}\right) \tag{14}$$

is the solution of the problem (10) with the condition $\widehat{u}_\alpha^\delta(\boldsymbol{\xi}, d) = \widehat{g}_\alpha^\delta(\boldsymbol{\xi})$.

The regularized solution to the problem (8) with a perturbed data is defined as the inverse Fourier transform (with respect to the first two variables) of $\widehat{v}_\alpha^\delta(\boldsymbol{\xi}, z)$

$$v_\alpha^\delta(\boldsymbol{\rho}, z) = \frac{1}{2\pi}\int\limits_{R^2} \widehat{g}_\alpha^\delta(\boldsymbol{\xi})\cosh\left((d-z)\sqrt{|\boldsymbol{\xi}|^2 - k^2}\right)e^{i\boldsymbol{\xi}\cdot\rho}d\boldsymbol{\xi}. \tag{15}$$

The parameter α plays the role the parameter of regularization which should depend on the error bound δ in order to get a convergence . The regularized solution for the exact data will be denoted by $v_\alpha(\boldsymbol{\rho}, z)$.

In [16] (lemmas 3.1, 3.2, Prop. 3.4) it was proved that the regularization method described above is convergent in the following sense:

Proposition 3.1 *Let v be the exact solution of (8) and let v_α^δ be the regularized solution (15) with noisy data g_δ such that $\|g - g_\delta\| \leq \delta$. Then*

$$\|v(\cdot, z) - v_\alpha(\cdot, z)\|_{L^2(\mathbb{R}^2)} \leq 2\|v(\cdot, 0)\|e^{-z\sqrt{\alpha - k^2}}, \tag{16}$$

$$\|v_\alpha(\cdot, z) - v_\alpha^\delta(\cdot, z)\| \leq \delta|e^{(d-z)|\sqrt{\alpha - k^2}}|. \tag{17}$$

If moreover, $\|v(\cdot, 0)\|_{L^2(\mathbb{R}^2)} \leq M$ for a-priori known constant M, $\delta \leq 2M$, and $\alpha = \alpha(\delta)$ is such that

$$\sqrt{\alpha - k^2} = -\frac{1}{d}\ln\frac{\delta}{2M}, \tag{18}$$

then for $z \in [0,d]$

$$\|v(\cdot,z) - v_\alpha^\delta(\cdot,z)\|_{L^2(\mathbb{R}^2)} \longrightarrow 0 \text{ as } \delta \longrightarrow 0, \tag{19}$$

and for $z \in (0,d]$

$$\|v(\cdot,z) - v_\alpha^\delta(\cdot,z)\|_{L^2(\mathbb{R}^2)} \leq \delta + 2M^{\frac{d-z}{d}}\delta^{\frac{z}{d}}. \tag{20}$$

4 Analysis of the model problem

Let us considered the model problem (5) introduced in Section 2. We look for solution in the space $H^2(\mathbb{R}^2) \times C^1(0,d)$. We assume that for the exact data the unique solution exists. The problem consists in a reconstruction of the solution in $\Omega = \Omega_0 \cup \Omega_1$ in the case of noisy data, ie. in the case when g_δ, $h_\delta \in L^2(\mathbb{R}^2)$ are given such that

$$\|g_\delta - g\|_{L^2(\mathbb{R}^2)} \leq \delta, \ \|h_\delta - h\|_{L^2(\mathbb{R}^2)} \leq \delta. \tag{21}$$

The solution of (5) can be represented by solutions of simpler problems with constant wavenumbers $k = k_1$ and $k = k_0$. Let w_r and v_r denote solutions of the following problems on $\Omega_1 = \mathbb{R} \times (s,d)$

$$\begin{cases} \Delta w + k_1^2 w = 0, & \text{in } \Omega_1 \\ w(\boldsymbol{\rho}, s) = 0 & \boldsymbol{\rho} \in \mathbb{R}^2, \\ \partial_z w(\boldsymbol{\rho}, d) = h(\boldsymbol{\rho}) & \boldsymbol{\rho} \in \mathbb{R}^2, \\ w(\cdot, z) \in L^2(\mathbb{R}^2), \end{cases} \tag{22}$$

$$\begin{cases} \Delta v + k_1^2 v = 0, & \text{in } \Omega_1 \\ v(\boldsymbol{\rho}, d) = g_0(\boldsymbol{\rho}) & \boldsymbol{\rho} \in \mathbb{R}^2, \\ \partial_z v(\boldsymbol{\rho}, d) = 0 & \boldsymbol{\rho} \in \mathbb{R}^2, \\ v(\cdot, z) \in L^2(\mathbb{R}^2), \end{cases} \tag{23}$$

where

$$g_0(\boldsymbol{\rho}) = g(\boldsymbol{\rho}) - w_r(\boldsymbol{\rho}, d). \tag{24}$$

Then, if solutions exist, $u|_{\Omega_1} = v_r + w_r$.

Similarly, let w_l and v_l be solutions of the following problems (25) and (26) on $\Omega_0 = \mathbb{R} \times (0, s)$:

$$\begin{cases} \Delta w + k_0^2 w = 0, & \text{in } \Omega_0 \\ w(\boldsymbol{\rho}, 0) = 0 & \boldsymbol{\rho} \in \mathbb{R}^2, \\ \partial_z w(\boldsymbol{\rho}, s) = h_1(\boldsymbol{\rho}) & \boldsymbol{\rho} \in \mathbb{R}^2, \\ w(\cdot, z) \in L^2(\mathbb{R}^2), \end{cases} \tag{25}$$

$$\begin{cases} \Delta v + k_0^2 v = 0, & \text{in } \Omega_1 \\ v(\boldsymbol{\rho}, s) = g_1(\boldsymbol{\rho}) & \boldsymbol{\rho} \in \mathbb{R}^2, \\ \partial_z v(\boldsymbol{\rho}, s) = 0 & \boldsymbol{\rho} \in \mathbb{R}^2, \\ v(\cdot, z) \in L^2(\mathbb{R}^2). \end{cases} \tag{26}$$

If

$$h_1(\boldsymbol{\rho}) = \partial_z w_r(\boldsymbol{\rho}, s) + \partial_z v_r(\boldsymbol{\rho}, s), \tag{27}$$

$$g_1(\boldsymbol{\rho}) = v_r(\boldsymbol{\rho}, s) - w_l(\boldsymbol{\rho}, s), \tag{28}$$

then $u|_{\Omega_0} = v_l + w_l$ if solutions of (25) and (26) exist.

Lemma 4.1 *If $k_1 < \frac{\pi}{2(d-s)}$, then the problem (22) is well posed, i.e. $\forall h \in L^2(\mathbb{R}^2)$ $\exists w_r \in H^2(\Omega_1)$ and*

$$\|w_r(\cdot, z)\|_{L^2(\mathbb{R}^2)} \le C_1 \|h\|_{L^2(\mathbb{R}^2)}, \text{ for } z \in [s, d], \tag{29}$$

$$\|\partial_z w_r(\cdot, z)\|_{L^2(\mathbb{R}^2)} \le D_1 \|h\|_{L^2(\mathbb{R}^2)}, \text{ for } z \in [s, d], \tag{30}$$

where

$$C_1 = \max\{d - s, \frac{1}{k_1} \tan(d - s)k_1\},$$

$$D_1 = \frac{1}{\cos((d - s)k_1)}.$$

PROOF. The first part of Lemma follows immediately form [16] (Lemma 2.1) when the interval $(0, d)$ is replaced by (s, d). Now, let us observe that since $\widehat{w_r}(\boldsymbol{\xi}, \cdot)$ satisfies the boundary value problem: $\frac{\partial^2}{\partial z^2}\widehat{w}(\boldsymbol{\xi}, z) = (|\boldsymbol{\xi}|^2 - k_1^2)\widehat{w}(\boldsymbol{\xi}, z)$ for $z \in (s, d)$ and $\widehat{w}(\boldsymbol{\xi}, s) = 0$, $\partial_z \widehat{w}(\boldsymbol{\xi}, s) = \widehat{h}(\boldsymbol{\xi})$ for $\boldsymbol{\xi} \in \mathbb{R}^2$, then

$$\partial_z \widehat{w_r}(\boldsymbol{\xi}, z) = \widehat{h}(\boldsymbol{\xi}) \frac{\cosh((z - s)\sqrt{|\boldsymbol{\xi}|^2 - k_1^2})}{\cosh((d - s)\sqrt{|\boldsymbol{\xi}|^2 - k_1^2})}. \tag{31}$$

Thus, taking into account that if $|\boldsymbol{\xi}| \geq k_1^2$ then

$$\cosh((z-s)\sqrt{|\boldsymbol{\xi}|^2 - k_1^2}) \leq \cosh((d-s)\sqrt{|\boldsymbol{\xi}|^2 - k_1^2}),$$

and if $|\boldsymbol{\xi}| < k_1^2$, then

$$\cosh((z-s)\sqrt{|\boldsymbol{\xi}|^2 - k_1^2}) = \cos((z-s)\sqrt{k_1^2 - |\boldsymbol{\xi}|^2})$$

and

$$(z-s)\sqrt{k_1^2 - |\boldsymbol{\xi}|^2} \leq (d-s)k_1 < \frac{\pi}{2}$$

we get

$$\|\partial_z w_r(\cdot, z)\| \leq \|h\| \frac{1}{\cosh((d-s)k_1)},$$

which gives (30). □

Remark 4.2 *From Lemma 4.1 it follows that if* $k_0 s < \frac{\pi}{2}$ *then the problem (25) is also well posed:* $\forall h_1 \in L^2(\mathbb{R}^2)$ $\exists w_l \in H^2(\Omega_0)$ *and*

$$\|w_l(\cdot, z)\|_{L^2(\mathbb{R}^2)} \leq C_2 \|h_1\|_{L^2(\mathbb{R}^2)}, \text{ for } z \in [0, s] \tag{32}$$

where $C_2 = \max\{s, \frac{1}{k_0} \tan sk_0\}$. *Moreover*

$$\|\partial_z w_l(\cdot, z)\|_{L^2(\mathbb{R}^2)} \leq D_2 \|h_1\|_{L^2(\mathbb{R}^2)}, \text{ for } z \in [0, s], \tag{33}$$

with $D_2 = \frac{1}{\cos(sk_0)}$.

From the assumption on existence the unique solution u for the exact data g and h, and from the well posedness of (22) and (25), it follows that there also exist unique solutions v_r and v_l of the Cauchy problems (23), (26). Moreover, since $u(\cdot, z)$ and $u_z(\cdot, z)$ are continuous with respect to z for $z \in (0, d)$ (by the assumption), it follows that

$$\begin{cases} u(\cdot, s) = v_r(\cdot, s), \\ \partial_z u(\cdot, s) = \partial_z v_r(\cdot, s) + \partial_z w_r(\cdot, s). \end{cases} \tag{34}$$

The problems (23), (26) as well as the auxiliary problem (8) are ill-posed. For approximate solving each of the problems (23) and (26) the regularization in frequency space will be applied separately, based on the results of Section 3 for (8). In the domains Ω_1, Ω_0 different regularization parameters α_1 and α_0 will be used.

5 Regularization

In the case of noisy data g_δ and h_δ, approximate solutions of the problems (22), (23), (25), (26) will be denoted by w_r^δ, v_{r,α_1}^δ, w_l^δ, v_{l,α_0}^δ, respectively, corresponding to perturbed data:

$$\begin{cases} g_0^\delta := g_\delta + w_r^\delta(\cdot, d), \\ h_1^\delta(\alpha_1) := \partial_z w_r^\delta(\cdot, s) + \partial_z v_{r,\alpha_1}^\delta(\cdot, s), \\ g_1^\delta(\alpha_1) := v_{r,\alpha_1}^\delta(\cdot, s) - w_l^\delta(\cdot, s). \end{cases} \tag{35}$$

Thus, according to (14),

$$\widehat{v_{r,\alpha_1}^\delta}(\boldsymbol{\xi}, z) = \widehat{g_{0,\alpha_1}^\delta}(\boldsymbol{\xi}) \cosh\left((d-z)\sqrt{|\boldsymbol{\xi}|^2 - k^2}\right) \qquad \text{for } z \in (s, d), \tag{36}$$

$$\widehat{v_{l,\alpha_0}^\delta}(\boldsymbol{\xi}, z) = \widehat{g_{1,\alpha_0}^\delta}(\boldsymbol{\xi}) \cosh\left((d-z)\sqrt{|\boldsymbol{\xi}|^2 - k^2}\right) \qquad \text{for } z \in (0, s). \tag{37}$$

In order to apply regularization results from Section 3 we have to find error bound for the new boundary data given above.

Remark 5.1 *From Proposition 3.1 applied to the problem (26) it follows*

$$\|v_l(\cdot, z) - v_{l,\alpha_0}^\delta(\cdot, z)\| \le 2\|u(\cdot, 0)\||e^{-z\sqrt{\alpha_0 - k_0^2}}| + \epsilon(\alpha_1, \delta)|e^{(s-z)\sqrt{\alpha_0 - k_0^2}}|, \tag{38}$$

where $\epsilon(\alpha_1, \delta)$ is an error bound for $\|g_1 - g_1^\delta(\alpha_1)\|$

According to (35) and (28), we have:

$$\|g_1 - g_1^\delta(\alpha_1)\| \le \|v_r(\cdot, s) - v_{r,\alpha_1}^\delta(\cdot, s)\| + \|w_l(\cdot, s) - w_l^\delta(\cdot, s)\|. \tag{39}$$

Direct application of Proposition 3.1 to the problem (23) does not give error bound for $\|v_r(\cdot, z) - v_{r,\alpha_1}^\delta(\cdot, z)\|$ on the left end of the z-interval, ie. for $z = s$. Such an estimation can be obtain by using an additional information about the exact solution. By the assumption, $u(\cdot, z) \in H^2(\mathbb{R}^2)$ for $z \in (0, d)$, thus

$$\exists M_s < \infty, \text{ such that } \int_{\mathbb{R}^2} \left| \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x, y, s) \right|^2 dxdy \le M_s. \tag{40}$$

We will assume that the constant M_s is known.

Lemma 5.2 *If M_s is a constant for which (40) holds, and $\|g - g_\delta\| \le \delta$ then*

$$\|v_r(\cdot, s) - v^\delta_{r,\alpha_1}(\cdot, s)\| \le \epsilon_1(\alpha_1, \delta), \tag{41}$$

where

$$\epsilon_1(\alpha_1, \delta) = \frac{1}{\alpha_1} M_s + \delta(1 + C_1)|e^{(d-s)\sqrt{\alpha_1 - k_1^2}}|$$

and the constants C_1 is defined in Lemma 4.1.

PROOF. We have

$$\|v_r(\cdot, s) - v_{r,\alpha_1}(\cdot, s)\|^2 = \int\limits_{|\boldsymbol{\xi}|^2 > \alpha_1} |\widehat{v_r}(\boldsymbol{\xi}, s)|^2 d\boldsymbol{\xi} \le \frac{1}{\alpha_1^2} \int\limits_{|\boldsymbol{\xi}|^2 > \alpha_1} \left| |\boldsymbol{\xi}|^2 \widehat{v_r}(\boldsymbol{\xi}, s) \right|^2 d\boldsymbol{\xi}.$$

Since $v_r(\boldsymbol{\rho}, s) = u(\boldsymbol{\rho}, s)$,

$$\int\limits_{|\boldsymbol{\xi}|^2 > \alpha_1} \left| |\boldsymbol{\xi}|^2 \widehat{v_r}(\boldsymbol{\xi}, s) \right|^2 d\boldsymbol{\xi} = \int\limits_{|\boldsymbol{\xi}|^2 > \alpha_1} \left| |\boldsymbol{\xi}|^2 \widehat{u}(\boldsymbol{\xi}, s) \right|^2 d\boldsymbol{\xi} \le M_s, \tag{42}$$

by the assumption. On the other side, applying Proposition 3.1 to the problem (23) and taking into account the estimation (17) we get

$$\|v_{r,\alpha_1}(\cdot, s) - v^\delta_{r,\alpha_1}(\cdot, s)\| \le \|g_0 - g_0^\delta\| |e^{(d-s)\sqrt{\alpha_1 - k_1^2}}|. \tag{43}$$

From the formula (24) and the inequality (29) we easily find

$$\|g_0^\delta - g_0\|_{L^2(\mathbb{R}^2)} \le \delta(1 + C_1). \tag{44}$$

Lemma follows now from (42), (43) and (44). $\square$

From (27), (35) and Remark 4.2 the second term in the inequality (39) can be estimated as follows:

$$\begin{aligned} \|w_l(\cdot, s) - w_l^\delta(\cdot, s)\| \le & C_2 \|\partial_z w_r(\cdot, s) - \partial_z w_r^\delta(\cdot, s)\| + \\ & + C_2 \|\partial_z v_r(\cdot, s) - \partial_z v^\delta_{r,\alpha_1}(\cdot, s)\|. \end{aligned} \tag{45}$$

Moreover, from Lemma 4.1 and the assumption (21) it follows

$$\|\partial_z w_r(\cdot,s) - \partial_z w_r^\delta(\cdot,s)\| \le \delta D_1 \tag{46}$$

with the corresponding constant described in Lemma 4.1.

Lemma 5.3 *Let M_s be a constant for which (40) holds, and $\|g - g_\delta\| \le \delta$. If $k_1(d-s) \le \frac{\pi}{2}$ and $\alpha_1 > k_0^2 - \frac{\pi}{2}$, then*

$$\|\partial_z v_r(\cdot,s) - \partial_z v_{r,\alpha_1}^\delta(\cdot,s)\| \le \epsilon_2(\alpha_1,\delta), \tag{47}$$

where

$$\epsilon_2(\alpha_1,\delta) = C_{\alpha_1} M_s \frac{1}{\sqrt{\alpha_1}} + \delta D_{\alpha_1},$$

and the constants C_{α_1} and D_{α_1} are given by (50) and (53), respectively.

PROOF. Applying the explicite formulas for the Fourier transform F of the solution $v_r(\cdot,z)$, $v_{r,\alpha_1}(\cdot,z)$, $z \in (s,d)$ (cf. (12) and (14)) we get

$$\|\partial_z v_r(\cdot,s) - \partial_z v_{r,\alpha_1}(\cdot,s)\|^2_{L^2(\mathbb{R}^2)} =$$

$$= \int\limits_{|\boldsymbol{\xi}|^2>\alpha_1} \left| \widehat{g}(\boldsymbol{\xi}) \sqrt{|\boldsymbol{\xi}|^2 - k_1^2} \sinh((d-s)\sqrt{|\boldsymbol{\xi}|^2 - k_1^2}) \right|^2 d\boldsymbol{\xi} =$$

$$= \int\limits_{|\boldsymbol{\xi}|^2>\alpha_1} \left| \widehat{v_r}(\boldsymbol{\xi},s) \sqrt{|\boldsymbol{\xi}|^2 - k_1^2} \tanh((d-s)\sqrt{|\boldsymbol{\xi}|^2 - k_1^2}) \right|^2 d\boldsymbol{\xi}. \tag{48}$$

It holds for $|\boldsymbol{\xi}|^2 > k_1^2 - (\frac{\pi}{2(d-s)})^2$, since in this case $\cosh((d-s)\sqrt{|\boldsymbol{\xi}|^2 - k_1^2}) \neq 0$. Due to the assumption $k_1(d-s) < \frac{\pi}{2}$, it occurs for any $\boldsymbol{\xi} \in \mathbb{R}^2$. Thus we get

$$\|\partial_z v_r(\cdot,s) - \partial_z v_{r,\alpha}(\cdot,s)\|^2 \le C_{\alpha_1}^2 \int\limits_{|\boldsymbol{\xi}|^2>\alpha_1} |\widehat{v_r}(\boldsymbol{\xi},s)|\boldsymbol{\xi}||^2 \, d\boldsymbol{\xi}, \tag{49}$$

where

$$C_{\alpha_1} = \begin{cases} 1 & \text{if } \alpha_1 > k_1^2 \\ \tan((d-s)\sqrt{k_1^2 - \alpha_1}) & \text{if } \alpha_1 \le k_1^2. \end{cases} \tag{50}$$

From the assumption (40) it follows

$$\int\limits_{|\boldsymbol{\xi}|^2>\alpha_1} |\widehat{v_r}(\boldsymbol{\xi},s)|\boldsymbol{\xi}||^2\, d\boldsymbol{\xi} \leq \frac{1}{\alpha_1} \int\limits_{|\boldsymbol{\xi}|^2>\alpha_1} \left|\widehat{v_r}(\boldsymbol{\xi},s)|\boldsymbol{\xi}|^2\right|^2 d\boldsymbol{\xi} \leq \frac{1}{\alpha_1} M_s^2. \tag{51}$$

Now, it remains to consider an influence of perturbation of g. Formulas (11) and (14) applied to the equation (23) give

$$\begin{aligned} &\|\partial_z v_{r,\alpha_1}(\cdot,s) - \partial_z v^\delta_{r,\alpha_1}(\cdot,s)\|_{L^2(\mathbb{R}^2)} \leq \\ &\leq \delta \sup_{|\boldsymbol{\xi}|^2\leq\alpha_1} \left|\sqrt{|\boldsymbol{\xi}|^2-k_1^2}\sinh((d-s)\sqrt{|\boldsymbol{\xi}|^2-k_1^2})\right| \leq \delta D_{\alpha_1}, \end{aligned} \tag{52}$$

where

$$D_{\alpha_1} := \frac{1}{2}\max\{k_1,\sqrt{\alpha_1}\}|e^{(d-s)\sqrt{\alpha_1-k_1^2}}|. \tag{53}$$

Therefore, Lemma 5.3 follows from (49), (51) and (52). $\square$

Now, we are going to formulate the main result of this Section concerning convergence of regularized solutions.

Let $w_r,\ v_r,\ w_l,\ v_l$ be the exact solutions to the problems (22), (23), (25), (26), respectively, with boundary conditions given by (24), (27), (28). Let $v^\delta_{r,\alpha_1},\ v^\delta_{l,\alpha_0}$ be regularized solutions to (23), (26) with noisy data (35) and let $w_r^\delta,\ w_l^\delta$ be solutions of (22) and (25) with noisy data h_δ and $h_1^\delta(\alpha_1)$, respectively.

Proposition 5.4 *If the assumptions of Lemma 5.3 are satisfied, then for $z \in (0,s)$*

$$\|v_l(\cdot,z) - v^\delta_{l,\alpha_0}(\cdot,z)\| \leq 2\|u(\cdot,0)\||e^{-z\sqrt{\alpha_0-k_0^2}}| + \epsilon(\alpha_1,\delta)|e^{(s-z)\sqrt{\alpha_0-k_0^2}}|, \tag{54}$$

where

$$\begin{aligned} &\epsilon(\alpha_1,\delta) = \\ &= M_s\left(\frac{1}{\alpha_1} + \frac{1}{\sqrt{\alpha_1}}C_2C_{\alpha_1}\right) + \delta e^{(d-s)\sqrt{\alpha_1-k_1^2}}\left(1 + C_1 + \frac{1}{2}C_2\sqrt{\alpha_1}\right). \end{aligned} \tag{55}$$

If moreover, α_0 is a function of α_1 and δ such that

$$\sqrt{\alpha_0 - k_0^2} = -\frac{1}{s}\ln\frac{\epsilon(\alpha_1,\delta)}{2M_0}, \tag{56}$$

then for $z \in (0, s)$

$$\|v_l(\cdot, z) - v^{\delta}_{l,\alpha_0}(\cdot, z)\|_{L^2(\mathbb{R}^2)} \le CM_0^{\frac{s-z}{s}} \epsilon(\alpha_1, \delta)^{\frac{z}{s}}. \tag{57}$$

PROOF. According to Remark 5.1 we have to estimate $\|g_1 - g^{\delta}_1(\alpha_1)\|$. Taking into account the inequalities (39), (45), (46) and Lammas 5.2, 5.3, we easily find that

$$\|g_1 - g^{\delta}_1(\alpha_1)\| \le \epsilon(\alpha_1, \delta),$$

where $\epsilon(\alpha_1, \delta)$ is given by (55). Now, Proposition 5.4 follows immediately from Proposition 3.1. □

If u is the exact solution of the model problem (5), then

$$u(\cdot, z) = \begin{cases} v_r(\cdot, z) + w_r(\cdot, z) & \text{for } z \in [s, d], \\ v_l(\cdot, z) + w_l(\cdot, z) & \text{for } z \in [0, s], \end{cases}$$

and the boundary value problems defining w_r and w_l are well posed for sufficiently small z-intervals: $d - s < \frac{\pi}{2k_1}$ and $s < \frac{\pi}{2k_0}$, respectively (Lemma 4.1 and Remark 4.2). In the case of noisy data g_δ and h_δ the regularized solution to (5) is given by the formula

$$u^{\delta}_{\alpha}(\cdot, z) = \begin{cases} v^{\delta}_{r,\alpha_1}(\cdot, z) + w^{\delta}_r(\cdot, z) & \text{for } z \in [s, d], \\ v^{\delta}_{l,\alpha_0}(\cdot, z) + w^{\delta}_l(\cdot, z) & \text{for } z \in [0, s]. \end{cases}$$

From Propositions 3.1 and 5.4 we get the following conclusion:

Theorem 5.5 *Let the assumptions of Lemma 5.3 be satisfied and $s < \frac{\pi}{2k_0}$. If $\alpha_1(\delta)$ and $\alpha_0(\alpha_1, \delta)$ are chosen according to formulas:*

$$\sqrt{\alpha_1 - k_1^2} = -\frac{1}{d-s} \ln \frac{\delta}{2M_s},$$

and (56), respectively, then for $z \in [0, d]$

$$\|u(\cdot, z) - u^{\delta}_{\alpha}(\cdot, z)\|_{L^2(\mathbb{R}^2)} \to 0 \text{ when } \delta \to 0.$$

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