

Spectral multipliers for sub-Laplacians on amenable Lie groups with exponential volume growth

Michael Gnewuch ^{*}

Mathematisches Seminar, Christian-Albrechts-Universität Kiel, Ludewig-Meyn-Str.4, D-24098 Kiel, Germany e-mail: mig@numerik.uni-kiel.de

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1 Introduction

Let G be a Lie group and Δ a left invariant sub-Laplacian on G . If $L^2(G)$ denotes the space of square integrable functions with respect to the right invariant Haar measure on G , then Δ is a selfadjoint operator on $L^2(G)$. Therefore every bounded Borel function f on \mathbb{R} induces a continuous operator $f(\Delta)$ on $L^2(G)$.

It is now natural to ask, under which additional conditions on f the operator $f(\Delta)$ is necessarily bounded on $L^p(G)$, $p \neq 2$. In this case we call f an *L^p -multiplier for Δ* . For more background information and various multiplier theorems we refer to [1], [3], [2], [5], [10], [8] and the literature mentioned therein.

Here we focus our attention on amenable groups with exponential volume growth and continuous functions f with compact support. Our aim is to show for a reasonably large class of Lie groups and sub-Laplacians that a certain degree of differentiability of f is sufficient for $f(\Delta)$ to extend to a bounded operator on $L^p(G)$, i. e. that Δ has *differentiable L^p -functional calculus*.

That this is not true on any group with exponential growth (in contrast to the situation on Lie groups with polynomial growth, cf. [1]), was shown by M. Christ and D. Müller in [2]. They gave examples of sub-Laplacians Δ on solvable Lie groups, which are for any $p \neq 2$ of *holomorphic L^p -type*, i. e., there exists some non-isolated point λ in the L^2 -spectrum of Δ and an open complex neighbourhood U of λ in \mathbb{C} such that every continuous L^p -multiplier, which vanishes at infinity, extends holomorphically to U . (More recent articles dealing with this topic are [8] and [7].)

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Therefore it is interesting to study new classes of groups and sub-Laplacians, to find out whether they admit differentiable L^p -functional calculus or not.

In Section 2 of this article we consider compact extensions of a class of solvable Lie groups. We modify and extend some methods, which were used in [6], to show that any sub-Laplacian on these groups possesses differentiable L^p -functional calculus for each $p \in [1, \infty]$.

In Section 3 we turn to some semidirect products of the 3-dimensional Heisenberg group \mathbb{H}_1 and the real line and study distinguished sub-Laplacians thereon. In fact, up to some exceptional cases, the groups and operators here are treated in Section 2 as well, namely when K is chosen to be the trivial compact group $\{1_K\}$. But in Section 3 we use different methods (introduced in [5] and again employed in [10]) to derive differentiable L^p -functional calculus. From the quantitative point of view our results here are better than the results about these special sub-Laplacians in Section 2.

2 Compact extensions of solvable groups

2.1 Preliminaries

Let \mathfrak{n} be a real m -dimensional nilpotent Lie algebra, and let N be \mathfrak{n} , endowed with the Campbell-Hausdorff multiplication. Then, up to isomorphism, N is the uniquely determined connected and simply connected nilpotent Lie group whose Lie algebra is \mathfrak{n} . Although the exponential map \exp_N of N is in fact the identity on \mathfrak{n} , we will use the notation \exp_N to make a clear distinction between the levels of Lie group and Lie algebra.

Let D be a derivation on \mathfrak{n} with eigenvalues λ_i , $i = 1, \dots, q$, whose real parts ρ_i are all strictly positive (or all strictly negative). We define ρ to be the real part ρ_i , which has the smallest absolute value. The trace of D will be denoted by Q . If D is diagonalizable over the field of complex numbers, we shall say that D is *semisimple*.

Let $\theta : \mathbb{R} \rightarrow \text{Aut}(\mathfrak{n}) = \text{Aut}(N)$ be the group homomorphism defined by $\theta(s) = e^{sD}$. Thus we can consider the semidirect product $H := N \rtimes_{\theta} \mathbb{R}$.

Furthermore, let K be a connected compact Lie group and $\gamma : K \rightarrow \text{Aut}(H)$ a group homomorphism such that the mapping

$$H \times K \rightarrow H, (h, k) \mapsto \gamma(k)h$$

is analytic. The group of main interest in this section is $G := H \rtimes_{\gamma} K$.

The measures we would like to consider, are the Lebesgue measures on N and \mathbb{R} , dn and dr , as well as the biinvariant Haar measure dk on K . For simplicity we may assume $dk(K) = 1$. The Lie algebra of the group K is denoted by \mathfrak{k} .

A right invariant Haar measure on H is given by $d^r h := dn dr$, and the measure $d^l h := e^{-rQ} dn dr$ is left invariant. We shall denote by μ the modular factor $\mu(n, r) := e^{rQ}$. It is easy to verify that $d^r g := dn dr dk$ is a right invariant Haar measure on G . As K is compact, the modular

function m on G is given by $m(n, r, k) = \mu(n, r)$. Hence the left invariant Haar measure on G is of the form $d^l g := e^{-rQ} dn dr dk$. For $p \in [1, \infty]$ let $L^p(G) := L^p(G, d^l g)$.

Since the modular functions μ and m are not trivial, the groups H and G are of exponential volume growth (cf. [11], §IX.1).

Let $\mathcal{Y}_1, \dots, \mathcal{Y}_p$ be left invariant vector fields on G , which generate the Lie algebra \mathfrak{g} of G . We are interested in the sub-Laplacian

$$\Delta = - \sum_{j=1}^p \mathcal{Y}_j^2 \quad (1)$$

and its heat kernel $(\phi_z)_{z \in \mathcal{H}_r}$, where \mathcal{H}_r denotes the right open complex halfplane. The heat kernel is defined by $e^{-z\Delta} f = f * \phi_z$ for all $f \in C_c^\infty(G)$.

2.2 Results

In the situation described above the following two theorems hold:

Theorem 1. *For any $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that for each $s \in \mathbb{R}$*

$$\|\phi_{1+is}\|_{L^1(G)} \leq C_\varepsilon (1 + |s|)^{\frac{Q}{2p} + 2 + \varepsilon}. \quad (2)$$

If D is semisimple, there exists a $C_0 > 0$ such that for all $s \in \mathbb{R}$

$$\|\phi_{1+is}\|_{L^1(G)} \leq C_0 (1 + |s|)^{\frac{Q}{2p} + 2}. \quad (3)$$

Theorem 2. *Let $f \in C_c(\mathbb{R})$, $\kappa > \frac{Q}{2p} + \frac{5}{2}$ and $p \in [1, \infty]$. If f lies in the Sobolev space $H^\kappa(\mathbb{R})$, the operator $f(\Delta)$ extends to a bounded endomorphism on $L^p(G)$, given by convolution from the right with the $L^1(G)$ -function*

$$k_f := \frac{1}{2\pi} \int_{\mathbb{R}} (f \cdot \exp)^\wedge(\xi) \phi_{1-i\xi} d\xi.$$

Theorem 2 follows directly from Theorem 1: With Theorem 1 and the Cauchy-Schwartz inequality it is easily verified, that for $f \in H^\kappa(\mathbb{R})$ the function k_f is integrable on G . The Fourier inversion formula implies for all $\varphi \in L^p \cap L^2(G)$

$$f(\Delta)\varphi = \frac{1}{2\pi} \int_{\mathbb{R}} (f \cdot \exp)^\wedge(\xi) e^{-(1-i\xi)\Delta} \varphi d\xi,$$

so we obtain $f(\Delta)\varphi = \varphi * k_f$ and $\|f(\Delta)\varphi\|_{L^p(G)} \leq \|k_f\|_{L^1(G)} \|\varphi\|_{L^p(G)}$.

Remark 1. Consider the special case, where K is the trivial group $\{1_K\}$, N a stratified group and \mathbb{R} is acting on N by the natural dilations. More precisely, let V_i , $i = 1, \dots, q$, be vector spaces with $\mathfrak{n} = V_1 \oplus \dots \oplus V_q$ and $[V_i, V_j] = V_{i+j}$ (with the convention $V_l = \{0\}$ for $l > q$), and let $Dv_j = jv_j$ for each $v_j \in V_j$.

In this situation Theorem 1 and 2 were proved in [6] (with slight restrictions on the form of the considered sub-Laplacians). In the Section *Improvements and open problems* of that article W. Hebisch mentioned that his results can be extended to any semidirect product H , defined as in our preceding section. So in the case $K = \{1_K\}$ our proof of Theorem 1 serves as a rigorous verification of the statement made by W. Hebisch.

When K and γ are non-trivial our results are new.

2.3 Proof of Theorem 1

If all ρ_i are strictly negative, the mapping $\tau(n, r, k) = (n, -r, k)$ is a group isomorphism between G and $\tilde{G} := (N \rtimes_{\tilde{\theta}} \mathbb{R}) \rtimes_{\tilde{\gamma}} K$ with $\tilde{\theta}(r) = e^{r(-D)}$ and $\tilde{\gamma} = \tau \circ \gamma \circ \tau^{-1}$. The operator $\tilde{\Delta} := d\tau(\Delta)$ is a sub-Laplacian on \tilde{G} and its heat kernel is given by $\tilde{\phi}_z = \phi_z \circ \tau$, which implies $\|\phi_z\|_{L^1(G)} = \|\tilde{\phi}_z\|_{L^1(\tilde{G})}$.

So we just have to prove the case, where all real parts ρ_i , $i = 1, \dots, q$, are strictly positive.

We shall reduce the L^1 -estimate of the heat kernel to a weighted L^2 -estimate in Proposition 1. But previously, we have to define a reasonable weight function w and to prove two preparatory lemmas.

Let $|\cdot|_D$ be a *homogeneous norm on N with respect to D* , i. e., a continuous mapping $|\cdot|_D : N \rightarrow [0, \infty[$, which is smooth away from the origin and which fulfils the conditions $|x|_D = 0$ iff $x = 0$, $|-x|_D = |x|_D$ and $|e^{sD}x|_D = e^s|x|_D$. (Such a homogeneous norm exists iff all the ρ_i , $i = 1, \dots, q$, are strictly positive; see e. g. [4], §2.5.) F_s shall denote the compact smooth surface $\{n \in N : |n|_D = e^s\}$. The weight function $w : G \rightarrow [0, \infty[$ is defined by $w(n, r, k) = |n|_D^Q$.

We consider a left invariant Riemannian metric d on G . Let 1_G be the unit element in G . Then we define $d(g)$ to be $d(1_G, g)$ for any $g \in G$.

Lemma 1. *There is a constant $C > 0$ such that for all $g = (n, r, k) \in G$*

$$|n|_D \leq Ce^{Cd(g)} \quad \text{and} \quad |r| \leq C(1 + d(g)).$$

Proof. Let B_r denote the Riemannian ball in G with centre 1_G and radius r . As its closure \overline{B}_r is compact, there are $q, p \in \mathbb{N}$ with

$$\overline{B}_1 \subset (B_q \cap H) \times K \quad \text{and} \quad \gamma(K)(\overline{B}_q \cap H) \subset B_p \cap H.$$

If $g_0 = (h_0, k_0)$ is in B_j , there exist $g_i = (h_i, k_i) \in B_1$, $i = 1, \dots, j$, with

$$g_0 = g_1 \cdot \dots \cdot g_j = (h_1 \cdot \gamma(k_1)h_2 \cdot \dots \cdot \gamma(k_1 \cdot \dots \cdot k_{j-1})h_j, k_1 \cdot \dots \cdot k_j).$$

Thus $h_0 = (n_0, r_0)$ is in $(B_p \cap H)^j$, and we can find $h'_i = (n_i, r_i) \in B_p \cap H$, $i = 1, \dots, j$, with

$$h_0 = h'_1 \cdot \dots \cdot h'_j = (n_1 \cdot e^{r_1 D} n_2 \cdot \dots \cdot e^{(r_1 + \dots + r_{j-1})D} n_j, r_1 + \dots + r_j).$$

There is a $C > 0$ with $\overline{B}_p \cap H \subset \{(n, r) : |n|_D \leq C, |r| \leq C\}$, which implies $|r_0| \leq Cj$. For all $n', m' \in N$ we have $|n' \cdot m'|_D \leq M \cdot \max\{|n'|_D, |m'|_D\}$, where $M := \max\{|n \cdot m|_D : |n|_D, |m|_D \leq 1\}$. Hence

$$|n_0|_D \leq M^{j-1} \max\{|n_1|_D, \dots, e^{r_1+\dots+r_{j-1}}|n_j|_D\} \leq CM^{j-1}e^{C(j-1)}.$$

□

Lemma 2. *There exists a $C > 0$ such that for each $R > 0$*

$$\int_{d(g) < R} (1 + w(g))^{-1} d^r g \leq C(1 + R)^2. \quad (4)$$

Proof. Lemma 1 ensures the existence of a constant $C > 0$ independent of $g = (n, r, k)$ such that $|n|_D \leq Ce^{Cd(g)}$ and $2|r| \leq C(1 + d(g))$ are satisfied. That implies

$$\int_{d(g) < R} \frac{d^r g}{1 + w(g)} \leq C(1 + R) \int_{|n|_D \leq Ce^{CR}} \frac{dn}{1 + |n|_D^Q}.$$

For the sake of simplicity we confine our analysis to the situation, where F_0 can be parametrized up to a set with surface measure zero by one chart $\varphi : U \rightarrow \mathbb{R}^m = \mathfrak{n}$. Here U is an open subset in \mathcal{R}^{m-1} . Let $R' := (CR + \ln(C))/Q$. The mapping $e^{sD} \circ \varphi$ is a parametrization of F_s and

$$\Phi : U \times]-\infty, R'[\rightarrow \{n \in N \setminus \{0\} : |n|_D < e^{R'Q}\}, (u, s) \mapsto e^{sD}(\varphi(u))$$

is a diffeomorphism onto the range of Φ with Jacobian determinant

$$e^{sQ} |\det(\partial_u \varphi(u), D(\varphi(u)))|.$$

With a suitable $C_0 > 0$ we obtain

$$\begin{aligned} \int_{|n|_D \leq e^{R'Q}} \frac{dn}{1 + |n|_D^Q} &= \int_{-\infty}^{R'} \left(\int_U \frac{e^{sQ} |\det(\partial_u \varphi(u), D(\varphi(u)))|}{1 + e^{sQ}} du \right) ds \\ &= C_0 \ln(1 + e^{R'Q}). \end{aligned}$$

□

Proposition 1. *There exists a $C > 0$ such that for every $s \in \mathbb{R}$*

$$\|\phi_{1+is}\|_{L^1(G)} \leq C(1 + |s|)^2 (1 + \|w^{1/2} \phi_{1+is}\|). \quad (5)$$

Here and in the sequel, $\|\cdot\|$ shall denote the norm on $L^2(G) = L^2(G, d^r g)$.

Because of inequality 4, the argument from [5], p. 160 (or [6], p. 438 – 439) can be used to prove Proposition 1. Consequently, in order to prove Theorem 1 it suffices to verify the following proposition:

Proposition 2. *For given $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that for each $s \in \mathbb{R}$*

$$\|w^{1/2}\phi_{1+is}\| \leq C_\varepsilon(1 + |s|)^{\frac{Q}{2\rho} + \varepsilon}. \quad (6)$$

If D is semisimple, then there exists a $C_0 > 0$, independent of s , with

$$\|w^{1/2}\phi_{1+is}\| \leq C_0(1 + |s|)^{\frac{Q}{2\rho}}. \quad (7)$$

For the proof of Proposition 2 it is useful to consider a distinguished basis of the Lie algebra \mathfrak{g} . Let $\{\mathcal{X}_1, \dots, \mathcal{X}_m\}$ be a basis of \mathfrak{n} , $\mathcal{X}_0 = (0, 1, 0) \in \mathfrak{n} \times \mathbb{R} \times \mathfrak{k}$ and $\{\mathcal{X}_{-1}, \dots, \mathcal{X}_{-n}\}$ a basis of \mathfrak{k} . These Lie algebra elements induce left invariant vector fields on G by

$$\mathcal{X}_j f(g) = \frac{d}{dt} f(g \cdot \exp_G(t\mathcal{X}_j))|_{t=0}, \quad j = -n, \dots, 0, \dots, m,$$

which we will identify with the Lie algebra elements themselves. If $j = 0, \dots, m$, we can also consider left invariant vector fields \mathcal{X}_j^H on H , defined by

$$\mathcal{X}_j^H \varphi(h) = \frac{d}{dt} \varphi(h \cdot \exp_H(t\mathcal{X}_j))|_{t=0}.$$

Analogously, we define for $j = 1, \dots, m$ and a function ψ on N

$$\mathcal{X}_j^N \psi(n) = \frac{d}{dt} \psi(n \cdot \exp_N(t\mathcal{X}_j))|_{t=0}.$$

Then the vector fields \mathcal{X}_j^H , $j = 0, \dots, m$, on H are given by

$$\mathcal{X}_0^H = \partial_r \quad \text{and} \quad \mathcal{X}_i^H = (e^{rD} \mathcal{X}_i)^N \quad \text{for all } i = 1, \dots, m. \quad (8)$$

For a given $k \in K$ let now $\tilde{\gamma}(k)$ be the uniquely determined linear mapping, which ensures commutativity in the diagram below:

$$\begin{array}{ccc} H & \xrightarrow{\gamma(k)} & H \\ \exp_H \uparrow & & \uparrow \exp_H \\ \mathfrak{h} & \xrightarrow{\tilde{\gamma}(k)} & \mathfrak{h} \end{array}$$

(A maybe more common notation for $\tilde{\gamma}(k)$ would be $d\gamma(k)$.) If $\tilde{\gamma}(k)$ will be represented as a matrix in the sequel, this is always meant with respect to the basis $\{\mathcal{X}_0, \dots, \mathcal{X}_m\}$ of \mathfrak{h} . With this convention we obtain for each $j \in \{0, \dots, m\}$ and $n \in N \setminus \{0\}$, $r \in \mathbb{R}$, $k \in K$

$$(\mathcal{X}_j w)(n, r, k) = \sum_{i=1}^m \tilde{\gamma}(k)_{i,j} [(e^{rD} \mathcal{X}_i)^N w](n). \quad (9)$$

The following statement can be calculated directly by transforming D into complex Jordan normal form: There are functions $P_{i,l,\nu} : \mathbb{R} \rightarrow \mathbb{C}$,

$i, l = 1, \dots, m; \nu = 1, \dots, q$, and constants $C, \mu > 0$ independent of $s \in \mathbb{R}$, satisfying

$$e^{sD} \mathcal{X}_i = \sum_{l=1}^m \left(\sum_{\nu=1}^q e^{s\rho_\nu} P_{i,l,\nu}(s) \right) \mathcal{X}_l \quad \text{and} \quad |P_{i,l,\nu}(s)| \leq C(1 + |s|^\mu). \quad (10)$$

If D is semisimple, each $P_{i,l,\nu}$ can be chosen as a bounded function.

Now we are going to establish a proposition, which is essential for our approach to the proof of Proposition 2:

Proposition 3. *For any $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that for all $i \in \{1, \dots, p\}$, $n \in N \setminus \{0\}$, $r \in \mathbb{R}$ and $k \in K$ the following inequality holds:*

$$|\mathcal{Y}_i w|(n, r, k) \leq C_\varepsilon \sum_{\nu=1}^q \left\{ e^{r(\rho_\nu + \varepsilon)} w(n)^{\frac{Q - \rho_\nu - \varepsilon}{Q}} + e^{r(\rho_\nu - \varepsilon)} w(n)^{\frac{Q - \rho_\nu + \varepsilon}{Q}} \right\}. \quad (11)$$

If D is semisimple, one can estimate

$$|\mathcal{Y}_i w|(n, r, k) \leq C_0 \sum_{\nu=1}^q e^{r\rho_\nu} w(n)^{\frac{Q - \rho_\nu}{Q}}, \quad (12)$$

with a constant $C_0 > 0$ independent of i, n, r and k .

Proof. For $i = -n, \dots, -1$ the functions $\mathcal{X}_i w$ are identically zero, as w does not depend on the variable k .

Using formula (10), we get for $n \in F_0$ and $i = \{1, \dots, m\}$

$$\begin{aligned} [(e^{rD} \mathcal{X}_i)^N w](e^{sD} n) &= e^{sQ} \frac{d}{dt} w(n \cdot \exp_N(te^{(r-s)D} \mathcal{X}_i))|_{t=0} \\ &= e^{sQ} (e^{(r-s)D} \mathcal{X}_i)^N w(n) = \sum_{\nu=1}^q e^{sQ + (r-s)\rho_\nu} \sum_{l=1}^m P_{i,l,\nu}(r-s) (\mathcal{X}_l^N w)(n). \end{aligned}$$

Hence there exists for any $\varepsilon > 0$ a constant $c_\varepsilon > 0$, fulfilling

$$|(e^{rD} \mathcal{X}_i)^N w|(e^{sD} n) \leq c_\varepsilon \sum_{\nu=1}^q e^{sQ + (r-s)\rho_\nu + |r-s|\varepsilon} \sum_{l=1}^m |\mathcal{X}_l^N w|(n). \quad (13)$$

Now define $C := c_\varepsilon \max\{\sum_{l=1}^m |(\mathcal{X}_l^N w)|(n) : n \in F_0\}$. Because of $w|_{F_0} \equiv 1$, there holds for $n \in F_0, r \in \mathbb{R}$

$$\begin{aligned} & |(e^{rD} \mathcal{X}_i)^N w|(e^{sD} n) \\ & \leq C e^{sQ} \sum_{\nu=1}^q \left\{ e^{(r-s)(\rho_\nu + \varepsilon)} w(n)^{\frac{Q - \rho_\nu - \varepsilon}{Q}} + e^{(r-s)(\rho_\nu - \varepsilon)} w(n)^{\frac{Q - \rho_\nu + \varepsilon}{Q}} \right\} \\ & = C \sum_{\nu=1}^q \left\{ e^{r(\rho_\nu + \varepsilon)} w(e^{sD} n)^{\frac{Q - \rho_\nu - \varepsilon}{Q}} + e^{r(\rho_\nu - \varepsilon)} w(e^{sD} n)^{\frac{Q - \rho_\nu + \varepsilon}{Q}} \right\}. \end{aligned} \quad (14)$$

We have $N \setminus \{0\} = \{e^{sD}F_0 : s \in \mathbb{R}\}$. Furthermore, the functions $\tilde{\gamma}_{i,j}$ are bounded on K and the \mathcal{Y}_i s are linear combinations of the \mathcal{X}_j s. Thus formulae (9) and (14) imply inequality (11). If D is semisimple, (13) can be simplified to

$$|(e^{rD}\mathcal{X}_i)^N w|(e^{sD}n) \leq c_0 \sum_{\nu=1}^q e^{sQ+(r-s)\rho_\nu} \sum_{l=1}^m |\mathcal{X}_l^N w|(n)$$

with a suitable constant $c_0 > 0$. Therefore inequality (12) follows. \square

To simplify the proof of Proposition 2, we state two preliminary lemmas:

Lemma 3. *For any $\delta > 0$ there exists a $C > 0$ such that for each $j \in \{1, \dots, p\}$ and each $z \in \mathbb{C}$ with $\Re(z) \geq \delta$ the inequality $\|e^{r\frac{Q}{2}}\mathcal{Y}_j\phi_z\| \leq C$ holds; here $e^{r\frac{Q}{2}}$ denotes the function $(n, r, k) \mapsto e^{r\frac{Q}{2}}$.*

Proof. Let $z \in \mathbb{C}$ with $\Re(z) \geq \delta$. If \mathcal{Y}_j has the form $\mathcal{Y}_j = \sum_{i=-n}^m a_{i,j}\mathcal{X}_i$ with $a_{i,j} \in \mathbb{R}$, then, by using the notation $\eta_j := \sum_{i=0}^m a_{i,j}\tilde{\gamma}_{0,i}$, we get

$$\langle \Delta\phi_z, e^{rQ}\phi_z \rangle = \sum_{j=1}^p (\|e^{r\frac{Q}{2}}\mathcal{Y}_j\phi_z\|^2 + \langle \mathcal{Y}_j\phi_z, \eta_j Q e^{rQ}\phi_z \rangle).$$

The Cauchy-Schwarz inequality and the fact that ϕ_z solves the homogeneous heat equation with respect to Δ imply

$$\begin{aligned} \sum_{j=1}^p \|e^{r\frac{Q}{2}}\mathcal{Y}_j\phi_z\|^2 &\leq \langle e^{r\frac{Q}{2}}\Delta\phi_z, e^{r\frac{Q}{2}}\phi_z \rangle + \sum_{j=1}^p |\eta_j|_\infty Q \langle |e^{r\frac{Q}{2}}\mathcal{Y}_j\phi_z|, |e^{r\frac{Q}{2}}\phi_z| \rangle \\ &\leq \|e^{r\frac{Q}{2}}\partial_z\phi_z\| \|e^{r\frac{Q}{2}}\phi_z\| + \sum_{j=1}^p \frac{1}{2} (\|e^{r\frac{Q}{2}}\mathcal{Y}_j\phi_z\|^2 + |\eta_j|_\infty^2 Q^2 \|e^{r\frac{Q}{2}}\phi_z\|^2). \end{aligned}$$

From $(e^{-z\Delta})^* = e^{-z^*\Delta}$ follows $\phi_z(g^{-1}) = m(g)\phi_z(g)$. As the modular function m is given by e^{rQ} , we get $\|e^{r\frac{Q}{2}}\phi_z\| = \|\phi_z\|$. Since $\phi_z = e^{-(z-\delta)\Delta}\phi_\delta$, $\|e^{r\frac{Q}{2}}\phi_z\| \leq \|\phi_\delta\|$ holds. By using Cauchy's formula, it is easy to verify that

$$\|e^{r\frac{Q}{2}}\partial_z\phi_z\| \leq \frac{2}{\delta} \sup\{\|e^{r\frac{Q}{2}}\phi_\zeta\| : |z - \zeta| < \frac{\delta}{2}\}.$$

\square

Lemma 4. *Let $j \in \{1, \dots, p\}$, $\delta > 0$, $\eta > 0$ and $\tilde{C} > 0$.*

(i) *If $Q \geq 2\eta$, there exists a $C > 0$ such that for $\alpha > 0$ and z with $\Re(z) \geq \delta$*

$$\langle |\mathcal{Y}_j\phi_z|, e^{r\eta} w^{\frac{Q-\eta}{Q}} |\phi_z| \rangle \leq \frac{\alpha}{\tilde{C}} \|w^{1/2}\mathcal{Y}_j\phi_z\|^2 + \frac{C}{\alpha} \|w^{1/2}\phi_z\|^{\frac{2Q-4\eta}{Q}}. \quad (15)$$

(ii) *If $Q < 2\eta < 2Q$, there exists a $C > 0$ with*

$$\langle |\mathcal{Y}_j\phi_z|, e^{r\eta} w^{\frac{Q-\eta}{Q}} |\phi_z| \rangle \leq \frac{\alpha}{\tilde{C}} \|w^{1/2}\mathcal{Y}_j\phi_z\|^2 + C\alpha^{\frac{\eta-Q}{\eta}} \quad (16)$$

for all $\alpha > 0$ and z with $\Re(z) \geq \delta$.

Proof. Consider $\alpha, \tilde{C}, \eta, \delta > 0$. Let $j \in \{1, \dots, p\}$ and z with $\Re(z) \geq \delta$.

(i) $Q \geq 2\eta$ implies

$$\langle |\mathcal{Y}_j \phi_z|, e^{r\eta} w^{\frac{Q-\eta}{Q}} |\phi_z| \rangle \leq \frac{\alpha}{\tilde{C}} \|w^{1/2} \mathcal{Y}_j \phi_z\|^2 + \frac{\tilde{C}}{\alpha} \|e^{r\eta} w^{\frac{Q-2\eta}{2Q}} \phi_z\|^2.$$

By using Hölder's inequality, one gets

$$\|e^{r\eta} w^{\frac{Q-2\eta}{2Q}} \phi_z\| \leq \|e^{r\frac{Q}{2}} \phi_z\|^{\frac{2\eta}{Q}} \|w^{1/2} \phi_z\|^{\frac{Q-2\eta}{Q}} \leq \|\phi_\delta\|^{\frac{2\eta}{Q}} \|w^{1/2} \phi_z\|^{\frac{Q-2\eta}{Q}}.$$

(ii) Let $Q < 2\eta < 2Q$. By using Hölder's inequality with exponents $p = Q/2(Q - \eta)$ and $p' = Q/(2\eta - Q)$, we can estimate

$$\begin{aligned} \langle |\mathcal{Y}_j \phi_z|, e^{r\eta} w^{\frac{Q-\eta}{Q}} |\phi_z| \rangle &\leq \|e^{r\frac{Q}{2}} \phi_z\| \|e^{r(\eta-\frac{Q}{2})} w^{\frac{Q-\eta}{Q}} \mathcal{Y}_j \phi_z\| \\ &\leq \|\phi_\delta\| \|e^{r(\eta-\frac{Q}{2})} w^{\frac{Q-\eta}{Q}} \mathcal{Y}_j \phi_z\| \leq \|\phi_\delta\| \|w^{1/2} \mathcal{Y}_j \phi_z\|^{\frac{2(Q-\eta)}{Q}} \|e^{r\frac{Q}{2}} \mathcal{Y}_j \phi_z\|^{\frac{2\eta-Q}{Q}}. \end{aligned}$$

As there exists a constant $C > 0$ with $\|e^{r\frac{Q}{2}} \mathcal{Y}_j \phi_z\| \leq C$ (cf. Lemma 3) and as the inequality $ab \leq a^r + b^{r'}$, $r' = \frac{r}{r-1}$ holds for all $a, b > 0$ and $r > 1$, we obtain inequality (16) with $r = Q/(Q - \eta)$ and a suitable $C > 0$. \square

Proof of Proposition 2. Lemma 1 states the existence of a $C > 0$ with $w(g) \leq C e^{Cd(g)}$, and $(e^{-z\Delta})_{\Re(z) > 0}$ is a holomorphic semigroup of operators on each weighted L^2 -space $L^2(G, e^{sd(g)} d^r g)$, $s \in \mathbb{R}$ (cf. [5], Lemma 1.2). Hence $z \mapsto w^{1/2} \phi_z$ is a holomorphic mapping from the right open complex halfplane into $L^2(G)$. Therefore there exists a $C > 0$ such that $\|w^{1/2} \phi_{1+is}\| \leq C$ for each $s \in [0, 1]$. Since $\phi_{1-is} = (\phi_{1+is})^*$, we have to consider only the case where $s \geq 1$. For $0 < \alpha \leq 1$ we define

$$\psi_\alpha(s) := \|w^{1/2} \phi_{\frac{1}{2}+(i+\alpha)s}\|^2.$$

Further we define $z := \frac{1}{2} + (i + \alpha)s$. Using this notation, we obtain

$$\begin{aligned} \partial_s \psi_\alpha(s) &= 2\Re\langle (i + \alpha)\partial_z \phi_z, w\phi_z \rangle = -2\Re\langle (i + \alpha)\Delta \phi_z, w\phi_z \rangle \\ &\leq 2 \sum_{j=1}^p \left(-\alpha \|w^{1/2} \mathcal{Y}_j \phi_z\|^2 + 2\langle |\mathcal{Y}_j \phi_z|, |\mathcal{Y}_j w|\phi_z \rangle \right). \end{aligned} \quad (17)$$

Case (1): $\dim \mathfrak{n} = 1$. Here, D is given by the 1×1 -matrix (Q) . Estimate (12) leads us to the inequality $|\mathcal{Y}_j w|(n, r, k) \leq C_0 e^{rQ}$ for $j \in \{1, \dots, p\}$. Hence

$$\langle |\mathcal{Y}_j \phi_z|, |\mathcal{Y}_j w|\phi_z \rangle \leq C_0 \|e^{r\frac{Q}{2}} \mathcal{Y}_j \phi_z\| \|e^{r\frac{Q}{2}} \phi_z\| \leq C,$$

with $C > 0$ independent of s, α (cf. Lemma 3). Thus there exists a $C > 0$ such that $\partial_s \psi_\alpha(s) \leq C$ for all s and α . That implies (with $\alpha := \frac{1}{2s}$)

$$\|w^{1/2} \phi_{1+is}\| \leq C \sqrt{1 + |s|} \quad \text{for } s \geq 1.$$

Case (2): $\dim \mathfrak{n} \geq 2$. For $j = 1, \dots, p$ and $\varepsilon > 0$ Proposition 3 implies

$$\begin{aligned} & \langle |\mathcal{Y}_j \phi_z|, |\mathcal{Y}_j w| |\phi_z| \rangle \\ & \leq C_\varepsilon \sum_{\nu=1}^q \langle |\mathcal{Y}_j \phi_z|, (e^{r(\rho_\nu + \varepsilon)} w^{\frac{Q - (\rho_\nu + \varepsilon)}{Q}} + e^{r(\rho_\nu - \varepsilon)} w^{\frac{Q - (\rho_\nu - \varepsilon)}{Q}}) |\phi_z| \rangle. \end{aligned} \quad (18)$$

(If D is semisimple, we can exchange ε by 0 in (18) and in the rest of this proof.) For every $\varepsilon < \rho$, $\rho_\nu + \varepsilon$ fulfills $Q > \rho_\nu + \varepsilon$ for any $\nu \in \{1, \dots, q\}$. According to (17), (18), (15) and (16) (with $C := 4qC_\varepsilon$), $\partial_s \psi_\alpha$ is majorized by a sum over $\eta \in \{\rho_\nu \pm \varepsilon : \nu = 1, \dots, q\}$, consisting of terms of the form

$$\frac{C}{\alpha} \|w^{1/2} \phi_z\|_{\frac{2Q-4\eta}{Q}} = \frac{C}{\alpha} \psi_\alpha^{\frac{Q-2\eta}{Q}} \quad \text{for } Q \geq 2\eta$$

and

$$C \alpha^{\frac{\eta-Q}{\eta}} \leq \frac{C}{\alpha} \quad \text{for } Q < 2\eta < 2Q.$$

Hence, there exists a $C > 0$ such that for all $\alpha \in]0, 1]$, $s \geq 1$ the function ψ_α is majorized by the solution u of the initial value problem

$$u'(s) = \frac{C}{\alpha} (1 + u(s))^{\frac{Q-2(\rho-\varepsilon)}{Q}}, \quad u(1) = \psi_\alpha(1),$$

which is given by

$$u(s) = \left(\frac{2(\rho-\varepsilon)C}{Q\alpha} (s-1) + (1 + \psi_\alpha(1))^{\frac{2(\rho-\varepsilon)}{Q}} \right)^{\frac{Q}{2(\rho-\varepsilon)}} - 1.$$

Hence, for $\alpha = \frac{1}{2s}$ there exists a constant $c_\varepsilon > 0$, independent of s , with

$$\|w^{1/2} \phi_{1+is}\| = \sqrt{\psi_\alpha(s)} \leq c_\varepsilon (1 + |s|)^{\frac{Q}{2(\rho-\varepsilon)}}.$$

□

3 Semidirect products of the Heisenberg group \mathbb{H}_1 and the real axis

3.1 Motivation

Noteworthy about Theorem 1 and 2 is, that the exponents in (2), (3) and the exponent κ in Theorem 2 tend to infinity with the ratio Q/ρ . There are indications that this phenomenon might be a consequence of our method of proof and does not reflect any underlying mathematical reality.

In [10] e. g., groups of the form $H = \mathbb{R}^2 \rtimes_\theta \mathbb{R}$ with $\theta(t) = e^{tD}$, D any 2×2 -matrix, were studied and for distinguished sub-Laplacians and their heat kernels the estimate

$$\|\phi_{1+i\xi}\|_{L^1(H)} \leq C(1 + |\xi|)^5 \quad (19)$$

was proven. (In some cases the estimates are better; the method which handled the most delicate case had been introduced in [5].) So the exponent of (19) is bounded, regardless of the action e^{tD} .

If we consider semidirect products $\mathbb{H}_1 \rtimes_{\theta} \mathbb{R}$ of the 3-dimensional Heisenberg group with the real axis, we can of course not expect such a result. The article [2] shows that not even all of this semidirect products admit differentiable L^p -functional calculus.

But if we confine ourselves to group homomorphisms θ , which are induced by derivations D in diagonal form with non-negative entries (or non-positive entries), we are able to derive an estimate like (19) with exponent 6 for all θ by transferring the methods from [5] and [10] to our situation.

3.2 Preliminaries

The *Heisenberg group* \mathbb{H}_1 is the set \mathbb{R}^3 endowed with the multiplication

$$(x, y, u)(x', y', u') = \left(x + x', y + y', u + u' + \frac{1}{2}(xy' - yx') \right).$$

The Lie algebra of \mathbb{H}_1 is the *Heisenberg algebra* \mathfrak{h}_1 .

Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta \geq 0$, and let D be the derivation on \mathfrak{h}_1 defined by $D(p, q, t) = (\alpha p, \beta q, (\alpha + \beta)t)$. The trace Q of D is then equal to $2(\alpha + \beta)$. Here our object of interest is the group $G := \mathbb{H}_1 \rtimes_{\theta} \mathbb{R}$, where $\theta(r) = e^{rD}$.

As in Section 2 the right invariant Haar measure $d^r g$ is simply the Lebesgue measure $dx dy du dr$ on \mathbb{R}^4 and the modular function is $m(g) = m(x, y, u, r) = e^{2(\alpha+\beta)r}$.

The left invariant vector fields on G , induced by the Lie algebra elements $\mathcal{X}_1 := (1, 0, 0, 0)$, $\mathcal{X}_2 := (0, 1, 0, 0)$, $\mathcal{X}_3 := (0, 0, 1, 0)$ and $\mathcal{X}_0 := (0, 0, 0, 1)$ from $\mathfrak{g} = \mathfrak{h}_1 \times \mathbb{R}$, are explicitly given by

$$\begin{aligned} \mathcal{X}_1 &= e^{\alpha r} \left(\partial_x - \frac{1}{2} y \partial_u \right), & \mathcal{X}_2 &= e^{\beta r} \left(\partial_y + \frac{1}{2} x \partial_u \right), \\ \mathcal{X}_3 &= e^{(\alpha+\beta)r} \partial_u, & \mathcal{X}_0 &= \partial_r. \end{aligned}$$

The operator $\Delta_S := -\sum_{j=0}^2 \mathcal{X}_j^2$ is a sub-Laplacian and $\Delta_L := -\sum_{j=0}^3 \mathcal{X}_j^2$ a full Laplacian on G . Let ϕ_t^S and ϕ_t^L denote the heat kernels of Δ_S and Δ_L , respectively. Further let $J(\Delta_S) := \{0, 1, 2\}$ and $J(\Delta_L) := \{0, 1, 2, 3\}$. In the sequel Δ shall denote the sub-Laplacian Δ_S as well as the full Laplacian Δ_L , and $(\phi_t)_{t>0}$ shall denote the heat kernel of Δ .

3.3 Results

Theorem 3. *There exists a $C > 0$ such that for each $\xi \in \mathbb{R}$ the inequality*

$$\|\phi_{1+i\xi}\|_{L^1(G)} \leq C(1 + |\xi|)^{\kappa} \quad (20)$$

holds; hereby we have

$$\kappa = \begin{cases} \frac{\alpha+\beta}{\min\{\alpha,\beta\}} + 2 & \text{if } \frac{\alpha}{\beta} \in [\frac{1}{3}, 3], \\ 6 & \text{otherwise.} \end{cases} \quad (21)$$

Like Theorem 1, Theorem 3 implies a multiplier theorem:

Theorem 4. *Let $p \in [1, \infty]$, $\varepsilon > 0$ and κ as in (21). Then each $f \in C_c \cap H^{\kappa+\frac{1}{2}+\varepsilon}(\mathbb{R})$ is an L^p -multiplier for Δ .*

Remark 2. (a) Extending the results of Theorem 3 and Theorem 4 to compact extensions $G \rtimes_{\gamma} K$ of G and sub-Laplacians $\Delta_K + d\gamma(\Delta)$, Δ_K a sub-Laplacian on K , is more or less trivial:

If $\alpha \neq \beta$, it can be shown that any homomorphism $\gamma : K \rightarrow \text{Aut}(G)$ has to be trivial, i. e., $\gamma(k)$ is the identity on G for any $k \in K$. (One can e. g. calculate the entries of the matrix $\tilde{\gamma} = d\gamma$ successively.) But then our sub-Laplacian is of the form $\Delta_K + \Delta$ and its heat kernel is given by $\phi_z^K \otimes \phi_z$, ϕ_z^K the heat kernel of Δ_K .

If $\alpha = \beta \neq 0$, the extension of the results is contained in Section 2.

If $\alpha = 0 = \beta$, then $G \rtimes_{\gamma} K$ has polynomial growth, so we refer to [1].

(b) If $\phi_z = \phi_z^S$ is the heat kernel of the sub-Laplacian and if $\alpha = \beta \neq 0$, Inequality (20) and Theorem 4 hold even with $\kappa = 3/2$ (cf. [9] or [4]).

3.4 Outline of the proof of Theorem 3

The general strategy for proving Theorem 3 is the same as in the proof of Theorem 1. That is, we want to reduce the L^1 -estimate of the heat kernel to a weighted L^2 -estimate. But this time we utilize a weight function w , which is independent of the action θ : We define $w : G \rightarrow \mathbb{R}$ by

$$w(x, y, u, r) = (1 + |x|)(1 + |y|)(1 + |u|).$$

Then there exists a $C > 0$ such that for any $R > 0$

$$\int_{d(g) \leq R} w(g)^{-1} d^r g \leq C(1 + R)^4.$$

Again, by using the same argument as in [5], we are able to find a constant $C > 0$ such that for each $\xi \in \mathbb{R}$

$$\|\phi_{1+i\xi}\|_{L^1(G)} \leq C(1 + |\xi|)^4(1 + \|w^{1/2}\phi_{1+i\xi}\|). \quad (22)$$

Therefore we are interested in estimating the term $|\partial_{\xi}\|w^{1/2}\phi_{1+i\xi}\|^2|$. We will do this step by step, beginning with weights of low order in x, y, u (like $|x|^{1/2}$ and $|y|^{1/2}$) and using the estimations of these terms to estimate higher order terms in x, y and u . We start with the analogue of Lemma 3:

Lemma 5. *For any $\delta > 0$ there exists a $C > 0$ such that for each $j \in J(\Delta)$ and each $z \in \mathbb{C}$ with $\Re(z) \geq \delta$ the inequality $\|e^{(\alpha+\beta)r}\mathcal{X}_j\phi_z\| \leq C$ holds.*

Lemma 5 can be verified easily by mimicking the proof of Lemma 3.

Lemma 6. *For any compact set $K \subset]0, \infty[$ there exists a $C > 0$ with*

$$\| |x|^{1/2} \phi_{\rho+i\xi} \| + \| |y|^{1/2} \phi_{\rho+i\xi} \| \leq C(1 + |\xi|)^{1/2} \quad (23)$$

and

$$\| |x|^{1/2} \mathcal{X}_j \phi_{\rho+i\xi} \| + \| |y|^{1/2} \mathcal{X}_j \phi_{\rho+i\xi} \| \leq C(1 + |\xi|)^{1/2} \quad (24)$$

for all $\rho \in K$, $\xi \in \mathbb{R}$ and all $j \in J(\Delta)$.

Proof. By using the notation $z = \rho + i\xi$ we get

$$\langle \Delta \phi_z, |y| \phi_z \rangle = \sum_{j \in J(\Delta)} \| |y|^{1/2} \mathcal{X}_j \phi_z \|^2 + \langle \mathcal{X}_2 \phi_z, e^{\beta r} \operatorname{sgn}(y) \phi_z \rangle.$$

On the one hand, from this and Lemma 5 there follows

$$\begin{aligned} |\partial_\xi \| |y|^{1/2} \phi_z \|^2| &= 2|\Re(i \langle \partial_z \phi_z, |y| \phi_z \rangle)| = 2|\Im \langle \Delta \phi_z, |y| \phi_z \rangle| \\ &\leq 2|\langle \mathcal{X}_2 \phi_z, e^{\beta r} \operatorname{sgn}(y) \phi_z \rangle| \leq 2\| \phi_z \| \| e^{\beta r} \mathcal{X}_2 \phi_z \| \leq C. \end{aligned}$$

We obtain $\| |y|^{1/2} \phi_z \|^2 \leq C(1 + |\xi|)$, because the mapping $K \ni \rho \mapsto \| |y|^{1/2} \phi_\rho \|^2$ is in particular continuous and thus bounded.

On the other hand, it follows from $(\partial_z + \Delta) \phi_z = 0$ and Cauchy's formula that

$$\sum_{j \in J(\Delta)} \| |y|^{1/2} \mathcal{X}_j \phi_z \|^2 \leq C + \| |y|^{1/2} \phi_z \| \| |y|^{1/2} \partial_z \phi_z \| \leq C(1 + |\xi|).$$

The rest of the statement can be obtained analogously. \square

Lemma 7. *For any compact set $K \subset]0, \infty[$ one can choose a $C > 0$ in such a way that for each $\rho \in K$, $\xi \in \mathbb{R}$ and each $j \in J(\Delta)$*

$$\| e^{(\frac{\alpha}{2} + \beta)r} |x|^{1/2} \phi_{\rho+i\xi} \| + \| e^{(\alpha + \frac{\beta}{2})r} |y|^{1/2} \phi_{\rho+i\xi} \| \leq C(1 + |\xi|)^{1/2} \quad (25)$$

and

$$\| e^{(\frac{\alpha}{2} + \beta)r} |x|^{1/2} \mathcal{X}_j \phi_{\rho+i\xi} \| + \| e^{(\alpha + \frac{\beta}{2})r} |y|^{1/2} \mathcal{X}_j \phi_{\rho+i\xi} \| \leq C(1 + |\xi|)^{1/2}. \quad (26)$$

Proof. Let again $z := \rho + i\xi$. With the notation $\gamma(r) = \exp((\alpha + \frac{\beta}{2})r)$ we get $\| \gamma |y|^{1/2} \phi_{\rho+i\xi} \| = \| |y|^{1/2} \phi_{\rho+i\xi} \|$, because of $\phi_z(g^{-1}) = m(g) \phi_z(g)$. Further

$$\begin{aligned} \langle \Delta \phi_z, \gamma^2 |y| \phi_z \rangle &= \sum_{j \in J(\Delta)} \| \gamma |y|^{1/2} \mathcal{X}_j \phi_z \|^2 + \langle \mathcal{X}_2 \phi_z, m \operatorname{sgn}(y) \phi_z \rangle \\ &\quad + \langle \mathcal{X}_0 \phi_z, (2\alpha + \beta) \gamma^2 |y| \phi_z \rangle. \end{aligned}$$

Here the absolute value of the last term can be majorized by

$$\frac{1}{2} \| \gamma |y|^{1/2} \mathcal{X}_0 \phi_z \|^2 + \frac{(2\alpha + \beta)^2}{2} \| \gamma |y|^{1/2} \phi_z \|^2.$$

Hence

$$\begin{aligned} \sum_{j \in J(\Delta)} \|\gamma|y|^{1/2} \mathcal{X}_j \phi_z\|^2 &\leq 2\|\gamma|y|^{1/2} \phi_z\| \|\gamma|y|^{1/2} \partial_z \phi_z\| + 2\|\sqrt{m} \phi_z\| \|\sqrt{m} \mathcal{X}_2 \phi_z\| \\ &+ (2\alpha + \beta)^2 \|\gamma|y|^{1/2} \phi_z\|^2 \leq C(1 + |\xi|). \end{aligned}$$

□

Lemma 8. *For any compact set $K \subset]0, \infty[$ there exists a $C > 0$ with*

$$\| |u|^{1/2} \phi_{\rho+i\xi} \| + \| |u|^{1/2} \mathcal{X}_j \phi_{\rho+i\xi} \| \leq C(1 + |\xi|) \quad (27)$$

for every $\rho \in K$, $\xi \in \mathbb{R}$ and $j \in J(\Delta)$.

Proof. We consider just the case $\Delta = \Delta_L$. (The proof for the heat kernel of the sub-Laplacian Δ_S is contained in the proof for the Laplacian Δ_L – one has just to ignore all terms in which \mathcal{X}_3 occurs.)

With $z = \rho + i\xi$ we get

$$\begin{aligned} \langle \Delta_L \phi_z^L, |u| \phi_z^L \rangle &= \sum_{j=0}^3 \| |u|^{1/2} \mathcal{X}_j \phi_z^L \|^2 - \frac{1}{2} \langle \mathcal{X}_1 \phi_z^L, e^{\alpha r} y \operatorname{sgn}(u) \phi_z^L \rangle \\ &+ \frac{1}{2} \langle \mathcal{X}_2 \phi_z^L, e^{\beta r} x \operatorname{sgn}(u) \phi_z^L \rangle + \langle \mathcal{X}_3 \phi_z^L, e^{(\alpha+\beta)r} \operatorname{sgn}(u) \phi_z^L \rangle. \end{aligned}$$

By proceeding as in the proof of (23) we obtain $|\partial_\xi| \| |u|^{1/2} \phi_z^L \|^2 \leq C(1 + |\xi|)$, and from this follows $\| |u|^{1/2} \phi_{\rho+i\xi}^L \|^2 \leq C(1 + |\xi|)^2$. As in the proof of (24) one derives eventually $\sum_{j=0}^3 \| |u|^{1/2} \mathcal{X}_j \phi_z^L \|^2 \leq C(1 + |\xi|)^2$. □

In Lemma 7 we got the same upper bound $C(1 + |\xi|^{1/2})$ for the terms $\| e^{(\alpha+\frac{\beta}{2})r} |y|^{1/2} \phi_{\rho+i\xi} \|$ and $\| e^{(\alpha+\frac{\beta}{2})r} |y|^{1/2} \mathcal{X}_j \phi_{\rho+i\xi} \|$ (which we shall call *related terms of* $\| |y|^{1/2} \phi_{\rho+i\xi} \|$) as for the terms $\| |y|^{1/2} \phi_{\rho+i\xi} \|$ and $\| |y|^{1/2} \mathcal{X}_j \phi_{\rho+i\xi} \|$ in Lemma 6. Similarly, we get from Lemma 8 the estimate

$$\| e^{\frac{\alpha+\beta}{2}r} |u|^{1/2} \phi_{\rho+i\xi} \| + \| e^{\frac{\alpha+\beta}{2}r} |u|^{1/2} \mathcal{X}_j \phi_{\rho+i\xi} \| \leq C(1 + |\xi|) \quad (28)$$

for the sum of the related terms of $\| |u|^{1/2} \phi_{\rho+i\xi} \|$. The last inequality holds again for any compact $K \subset]0, \infty[$, $\rho \in K$, $\xi \in \mathbb{R}$, $j \in J$ and the constant C depends just on K .

By using the same notation and just the same techniques that we have used so far, we obtain

$$\| x \phi_{\rho+i\xi} \| + \| y \phi_{\rho+i\xi} \| \leq C(1 + |\xi|) \quad (29)$$

and this upper bound holds also for the related terms $\| e^{\beta r} x \phi_{\rho+i\xi} \|$, $\| e^{\alpha r} y \phi_{\rho+i\xi} \|$, $\| e^{\beta r} x \mathcal{X}_j \phi_{\rho+i\xi} \|$ and $\| e^{\alpha r} y \mathcal{X}_j \phi_{\rho+i\xi} \|$.

After this it is easy to derive that $\| |xu|^{1/2} \phi_{\rho+i\xi} \|$, $\| |yu|^{1/2} \phi_{\rho+i\xi} \|$ and the related terms are bounded by $C(1 + |\xi|)^{3/2}$.

In a similar way one establishes

$$\| |x|y|^{1/2} \phi_{\rho+i\xi} \| + \| |x|^{1/2} y \phi_{\rho+i\xi} \| \leq C(1 + |\xi|)^{3/2} \quad (30)$$

(and this bound holds also for the related terms). With this bunch of estimates we are able to verify $\| w^{1/2} \phi_{1+i\xi} \| \leq C(1 + |\xi|)^2$. Therefore, with regard to inequality (22) and Theorem 1, Theorem 3 holds.

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