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Spectral multipliers for sub-Laplacians on amenable Lie groups with exponential volume growth

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1 Introduction

Let G be a Lie group and $\varDelta$ a left invariant sub-Laplacian on G. If $L^2(G)$ denotes the space of square integrable functions with respect to the right invariant Haar measure on G, then $\varDelta$ is a selfadjoint operator on $L^2(G)$. Therefore every bounded Borel function f on $\mathbb{R}$ induces a continuous operator $f(\varDelta)$ on $L^2(G)$.

It is now natural to ask, under which additional conditions on f the operator $f(\varDelta)$ is necessarily bounded on $L^p(G)$, $p \neq 2$. In this case we call f an *L^p-multiplier for $\varDelta$*. For more background information and various multiplier theorems we refer to [1], [3], [2], [5], [10], [8] and the literature mentioned therein.

Here we focus our attention on amenable groups with exponential volume growth and continuous functions f with compact support. Our aim is to show for a reasonably large class of Lie groups and sub-Laplacians that a certain degree of differentiability of f is sufficient for $f(\varDelta)$ to extend to a bounded operator on $L^p(G)$, i. e. that $\varDelta$ has *differentiable L^p-functional calculus*.

That this is not true on any group with exponential growth (in contrast to the situation on Lie groups with polynomial growth, cf. [1]), was shown by M. Christ and D. Müller in [2]. They gave examples of sub-Laplacians $\varDelta$ on solvable Lie groups, which are for any $p \neq 2$ of *holomorphic L^p-type*, i. e., there exists some non-isolated point λ in the L^2-spectrum of $\varDelta$ and an open complex neighbourhood U of λ in $\mathbb{C}$ such that every continuous L^p-multiplier, which vanishes at infinity, extends holomorphically to U. (More recent articles dealing with this topic are [8] and [7].)

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Therefore it is interesting to study new classes of groups and sub-Laplacians, to find out whether they admit differentiable L^p-functional calculus or not.

In Section 2 of this article we consider compact extensions of a class of solvable Lie groups. We modify and extend some methods, which were used in [6], to show that any sub-Laplacian on these groups possesses differentiable L^p-functional calculus for each $p \in [1, \infty]$.

In Section 3 we turn to some semidirect products of the 3-dimensional Heisenberg group $\mathbb{H}_1$ and the real line and study distinguished sub-Laplacians thereon. In fact, up to some exceptional cases, the groups and operators here are treated in Section 2 as well, namely when K is chosen to be the trivial compact group $\{1_K\}$. But in Section 3 we use different methods (introduced in [5] and again employed in [10]) to derive differentiable L^p-functional calculus. From the quantitative point of view our results here are better than the results about these special sub-Laplacians in Section 2.

2 Compact extensions of solvable groups

2.1 Preliminaries

Let $\mathfrak{n}$ be a real m-dimensional nilpotent Lie algebra, and let N be $\mathfrak{n}$, endowed with the Campbell-Hausdorff multiplication. Then, up to isomorphism, N is the uniquely determined connected and simply connected nilpotent Lie group whose Lie algebra is $\mathfrak{n}$. Although the exponential map $\exp_N$ of N is in fact the identity on $\mathfrak{n}$, we will use the notation $\exp_N$ to make a clear distinction between the levels of Lie group and Lie algebra.

Let D be a derivation on $\mathfrak{n}$ with eigenvalues λ_i, $i = 1, ..., q$, whose real parts ρ_i are all strictly positive (or all strictly negative). We define ρ to be the real part ρ_i, which has the smallest absolute value. The trace of D will be denoted by Q. If D is diagonalizable over the field of complex numbers, we shall say that D is *semisimple.*

Let $\theta : \mathbb{R} \to \mathrm{Aut}(\mathfrak{n}) = \mathrm{Aut(N)}$ be the group homomorphism defined by $\theta(s) = e^{sD}$. Thus we can consider the semidirect product $H := N \rtimes_\theta \mathbb{R}$.

Furthermore, let K be a connected compact Lie group and $\gamma : K \to \mathrm{Aut(H)}$ a group homomorphism such that the mapping

$$H \times K \to H \, , (h, k) \mapsto \gamma(k)h$$

is analytic. The group of main interest in this section is $G := H \rtimes_\gamma K$.

The measures we would like to consider, are the Lebesgue measures on N and $\mathbb{R}$, dn and dr, as well as the biinvariant Haar measure dk on K. For simplicity we may assume $dk(K) = 1$. The Lie algebra of the group K is denoted by $\mathfrak{k}$.

A right invariant Haar measure on H is given by $d^r h := dn\, dr$, and the measure $d^l h := e^{-rQ}\, dn\, dr$ is left invariant. We shall denote by μ the modular factor $\mu(n, r) := e^{rQ}$. It is easy to verify that $d^r g := dn\, dr\, dk$ is a right invariant Haar measure on G. As K is compact, the modular function m on G is given by $m(n,r,k) = \mu(n,r)$. Hence the left invariant Haar measure on G is of the form $d^l g := e^{-rQ}\, dn\, dr\, dk$. For $p \in [1,\infty]$ let $L^p(G) := L^p(G, d^r g)$.

Since the modular functions μ and m are not trivial, the groups H and G are of exponential volume growth (cf. [11], §IX.1).

Let $\mathcal{Y}_1, ..., \mathcal{Y}_p$ be left invariant vector fields on G, which generate the Lie algebra $\mathfrak{g}$ of G. We are interested in the sub-Laplacian

$$\Delta = -\sum_{j=1}^{p} \mathcal{Y}_j^2 \tag{1}$$

and its heat kernel $(\phi_z)_{z\in\mathcal{H}_r}$, where $\mathcal{H}_r$ denotes the right open complex halfplane. The heat kernel is defined by $e^{-z\Delta} f = f * \phi_z$ for all $f \in C_c^\infty(G)$.

2.2 Results

In the situation described above the following two theorems hold:

Theorem 1. *For any $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that for each $s \in \mathbb{R}$*

$$\|\phi_{1+is}\|_{L^1(G)} \le C_\varepsilon (1+|s|)^{\frac{Q}{2\rho}+2+\varepsilon} . \tag{2}$$

If D is semisimple, there exists a $C_0 > 0$ such that for all $s \in \mathbb{R}$

$$\|\phi_{1+is}\|_{L^1(G)} \le C_0 (1+|s|)^{\frac{Q}{2\rho}+2} . \tag{3}$$

Theorem 2. *Let $f \in C_c(\mathbb{R})$, $\kappa > \frac{Q}{2\rho} + \frac{5}{2}$ and $p \in [1,\infty]$. If f lies in the Sobolev space $H^\kappa(\mathbb{R})$, the operator $f(\Delta)$ extends to a bounded endomorphism on $L^p(G)$, given by convolution from the right with the $L^1(G)$-function*

$$k_f := \frac{1}{2\pi} \int_{\mathbb{R}} (f \cdot \exp)^\wedge(\xi) \phi_{1-i\xi}\, d\xi .$$

Theorem 2 follows directly from Theorem 1: With Theorem 1 and the Cauchy-Schwartz inequality it is easily verified, that for $f \in H^\kappa(\mathbb{R})$ the function k_f is integrable on G. The Fourier inversion formula implies for all $\varphi \in L^p \cap L^2(G)$

$$f(\Delta)\varphi = \frac{1}{2\pi} \int_{\mathbb{R}} (f \cdot \exp)^\wedge(\xi) e^{-(1-i\xi)\Delta} \varphi\, d\xi ,$$

so we obtain $f(\Delta)\varphi = \varphi * k_f$ and $\|f(\Delta)\varphi\|_{L^p(G)} \le \|k_f\|_{L^1(G)} \|\varphi\|_{L^p(G)}$.

Remark 1. Consider the special case, where K is the trivial group $\{1_K\}$, N a stratified group and $\mathbb{R}$ is acting on N by the natural dilations. More precisely, let V_i, $i = 1, ..., q$, be vector spaces with $\mathfrak{n} = V_1 \oplus ... \oplus V_q$ and $[V_i, V_j] = V_{i+j}$ (with the convention $V_l = \{0\}$ for $l > q$), and let $Dv_j = jv_j$ for each $v_j \in V_j$.

In this situation Theorem 1 and 2 were proved in [6] (with slight restrictions on the form of the considered sub-Laplacians). In the Section *Improvements and open problems* of that article W. Hebisch mentioned that his results can be extended to any semidirect product H, defined as in our preceding section. So in the case $K = \{1_K\}$ our proof of Theorem 1 serves as a rigorous verification of the statement made by W. Hebisch.

When K and γ are non-trivial our results are new.

2.3 Proof of Theorem 1

If all ρ_i are strictly negative, the mapping $\tau(n,r,k) = (n,-r,k)$ is a group isomorphism between G and $\tilde{G} := (N \rtimes_{\tilde{\theta}} \mathbb{R}) \rtimes_{\tilde{\gamma}} K$ with $\tilde{\theta}(r) = e^{r(-D)}$ and $\tilde{\gamma} = \tau \circ \gamma \circ \tau^{-1}$. The operator $\tilde{\Delta} := d\tau(\Delta)$ is a sub–Laplacian on $\tilde{G}$ and its heat kernel is given by $\tilde{\phi}_z = \phi_z \circ \tau$, which implies $\|\phi_z\|_{L^1(G)} = \|\tilde{\phi}_z\|_{L^1(\tilde{G})}$.

So we just have to prove the case, where all real parts ρ_i, $i = 1, ..., q$, are strictly positive.

We shall reduce the L^1-estimate of the heat kernel to a weighted L^2-estimate in Proposition 1. But previously, we have to define a reasonable weight function w and to prove two preparatory lemmas.

Let $|\cdot|_D$ be a *homogeneous norm on N with respect to D*, i. e., a continuous mapping $|\cdot|_D : N \to [0,\infty[$, which is smooth away from the origin and which fulfils the conditions $|x|_D = 0$ iff $x = 0$, $|-x|_D = |x|_D$ and $|e^{sD}x|_D = e^s|x|_D$. (Such a homogeneous norm exists iff all the ρ_i, $i = 1, ..., q$, are strictly positive; see e. g. [4], §2.5.) F_s shall denote the compact smooth surface $\{n \in N : |n|_D = e^s\}$. The weight function $w : G \to [0,\infty[$ is defined by $w(n,r,k) = |n|_D^Q$.

We consider a left invariant Riemannian metric d on G. Let 1_G be the unit element in G. Then we define $d(g)$ to be $d(1_G, g)$ for any $g \in G$.

Lemma 1. *There is a constant $C > 0$ such that for all $g = (n,r,k) \in G$*

$$|n|_D \le Ce^{Cd(g)} \quad \textit{and} \quad |r| \le C(1 + d(g))\,.$$

Proof. Let B_r denote the Riemannian ball in G with centre 1_G and radius r. As its closure $\overline{B}_r$ is compact, there are q, $p \in \mathbb{N}$ with

$$\overline{B}_1 \subset (B_q \cap H) \times K \quad \text{and} \quad \gamma(K)(\overline{B}_q \cap H) \subset B_p \cap H\,.$$

If $g_0 = (h_0, k_0)$ is in B_j, there exist $g_i = (h_i, k_i) \in B_1$, $i = 1, ..., j$, with

$$g_0 = g_1 \cdot ... \cdot g_j = (h_1 \cdot \gamma(k_1)h_2 \cdot ... \cdot \gamma(k_1 \cdot ... \cdot k_{j-1})h_j, k_1 \cdot ... \cdot k_j)\,.$$

Thus $h_0 = (n_0, r_0)$ is in $(B_p \cap H)^j$, and we can find $h'_i = (n_i, r_i) \in B_p \cap H$, $i = 1, ..., j$, with

$$h_0 = h'_1 \cdot ... \cdot h'_j = (n_1 \cdot e^{r_1 D}n_2 \cdot ... \cdot e^{(r_1 + ... + r_{j-1})D}n_j, r_1 + ... + r_j)\,.$$

There is a $C > 0$ with $\overline{B}_p \cap H \subset \{(n, r) : |n|_D \le C, |r| \le C\}$, which implies $|r_0| \le Cj$. For all n', $m' \in N$ we have $|n' \cdot m'|_D \le M \cdot \max\{|n'|_D, |m'|_D\}$, where $M := \max\{|n \cdot m|_D : |n|_D, |m|_D \le 1\}$. Hence

$$|n_0|_D \le M^{j-1} \max\{|n_1|_D, ..., e^{r_1+...+r_{j-1}}|n_j|_D\} \le CM^{j-1}e^{C(j-1)}\,.$$

□

Lemma 2. *There exists a $C > 0$ such that for each $R > 0$*

$$\int_{d(g)<R} (1 + w(g))^{-1} d^r g \le C(1 + R)^2\,. \tag{4}$$

Proof. Lemma 1 ensures the existence of a constant $C > 0$ independent of $g = (n, r, k)$ such that $|n|_D \le Ce^{Cd(g)}$ and $2|r| \le C(1 + d(g))$ are satisfied. That implies

$$\int\limits_{d(g)<R} \frac{d^r g}{1 + w(g)} \le C(1 + R) \int\limits_{|n|_D \le Ce^{CR}} \frac{dn}{1 + |n|_D^Q}\,.$$

For the sake of simplicity we confine our analysis to the situation, where F_0 can be parametrized up to a set with surface measure zero by one chart $\varphi : U \to \mathbb{R}^m = \mathfrak{n}$. Here U is an open subset in $\mathcal{R}^{m-1}$. Let $R' := (CR + \ln(C))/Q$. The mapping $e^{sD} \circ \varphi$ is a parametrization of F_s and

$$\Phi : U \times]-\infty, R'[\to \{n \in N \setminus \{0\} : |n|_D < e^{R'Q}\}\,, (u, s) \mapsto e^{sD}(\varphi(u))$$

is a diffeomorphism onto the range of Φ with Jacobian determinant

$$e^{sQ}|\det(\partial_u \varphi(u), D(\varphi(u)))|\,.$$

With a suitable $C_0 > 0$ we obtain

$$\begin{aligned}\int\limits_{|n|_D \le e^{R'Q}} \frac{dn}{1 + |n|_D^Q} &= \int\limits_{-\infty}^{R'} \Big(\int\limits_U \frac{e^{sQ}|\det(\partial_u \varphi(u), D(\varphi(u)))|}{1 + e^{sQ}}\, du \Big)\, ds \\ &= C_0 \ln(1 + e^{R'Q})\,.\end{aligned}$$

□

Proposition 1. *There exists a $C > 0$ such that for every $s \in \mathbb{R}$*

$$\|\phi_{1+is}\|_{L^1(G)} \le C(1 + |s|)^2(1 + \|w^{1/2}\phi_{1+is}\|)\,. \tag{5}$$

Here and in the sequel, $\|\cdot\|$ shall denote the norm on $L^2(G) = L^2(G, d^r g)$.

Because of inequality 4, the argument from [5], p. 160 (or [6], p. 438 – 439) can be used to prove Proposition 1. Consequently, in order to prove Theorem 1 it suffices to verify the following proposition:

Proposition 2. *For given $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that for each $s \in \mathbb{R}$*

$$\|w^{1/2}\phi_{1+is}\| \le C_\varepsilon(1+|s|)^{\frac{Q}{2\rho}+\varepsilon} \,. \tag{6}$$

If D is semisimple, then there exists a $C_0 > 0$, independent of s, with

$$\|w^{1/2}\phi_{1+is}\| \le C_0(1+|s|)^{\frac{Q}{2\rho}} \,. \tag{7}$$

For the proof of Proposition 2 it is useful to consider a distinguished basis of the Lie algebra $\mathfrak{g}$. Let $\{\mathcal{X}_1, ..., \mathcal{X}_m\}$ be a basis of $\mathfrak{n}$, $\mathcal{X}_0 = (0,1,0) \in \mathfrak{n} \times \mathbb{R} \times \mathfrak{k}$ and $\{\mathcal{X}_{-1}, ..., \mathcal{X}_{-n}\}$ a basis of $\mathfrak{k}$. These Lie algebra elements induce left invariant vector fields on G by

$$\mathcal{X}_j f(g) = \frac{d}{dt} f(g \cdot \exp_G(t\mathcal{X}_j))|_{t=0} \,, \quad j = -n, ..., 0, ..., m \,,$$

which we will identify with the Lie algebra elements themselves. If $j = 0, ..., m$, we can also consider left invariant vector fields $\mathcal{X}_j^H$ on H, defined by

$$\mathcal{X}_j^H \varphi(h) = \frac{d}{dt}\varphi(h \cdot \exp_H(t\mathcal{X}_j))|_{t=0} \,.$$

Analogously, we define for $j = 1, ..., m$ and a function ψ on N

$$\mathcal{X}_j^N \psi(n) = \frac{d}{dt}\psi(n \cdot \exp_N(t\mathcal{X}_j))|_{t=0} \,.$$

Then the vector fields $\mathcal{X}_j^H$, $j = 0, ..., m$, on H are given by

$$\mathcal{X}_0^H = \partial_r \quad \text{and} \quad \mathcal{X}_i^H = (e^{rD}\mathcal{X}_i)^N \quad \text{for all } i = 1, ..., m. \tag{8}$$

For a given $k \in K$ let now $\tilde\gamma(k)$ be the uniquely determined linear mapping, which ensures commutativity in the diagram below:

$$\begin{array}{ccc} H & \xrightarrow{\gamma(k)} & H \\ {\scriptstyle \exp_H}\uparrow & & \uparrow{\scriptstyle \exp_H} \\ \mathfrak{h} & \xrightarrow[\tilde\gamma(k)]{} & \mathfrak{h} \end{array}$$

(A maybe more common notation for $\tilde\gamma(k)$ would be $d\gamma(k)$.) If $\tilde\gamma(k)$ will be represented as a matrix in the sequel, this is always meant with respect to the basis $\{\mathcal{X}_0, ..., \mathcal{X}_m\}$ of $\mathfrak{h}$. With this convention we obtain for each $j \in \{0, ..., m\}$ and $n \in N \setminus \{0\}$, $r \in \mathbb{R}$, $k \in K$

$$(\mathcal{X}_j w)(n, r, k) = \sum_{i=1}^{m} \tilde\gamma(k)_{i,j}[(e^{rD}\mathcal{X}_i)^N w](n) \,. \tag{9}$$

The following statement can be calculated directly by transforming D into complex Jordan normal form: There are functions $P_{i,l,\nu} : \mathbb{R} \to \mathbb{C}$,

$i, l = 1, ..., m$; $\nu = 1, ..., q$, and constants C, $\mu > 0$ independent of $s \in \mathbb{R}$, satisfying

$$e^{sD}\mathcal{X}_i = \sum_{l=1}^{m} \Big(\sum_{\nu=1}^{q} e^{s\rho_\nu} P_{i,l,\nu}(s) \Big) \mathcal{X}_l \quad \textit{and} \quad |P_{i,l,\nu}(s)| \le C(1 + |s|^\mu)\,. \tag{10}$$

If D is semisimple, each $P_{i,l,\nu}$ can be chosen as a bounded function.

Now we are going to establish a proposition, which is essential for our approach to the proof of Proposition 2:

Proposition 3. *For any $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that for all $i \in \{1, ..., p\}$, $n \in N \setminus \{0\}$, $r \in \mathbb{R}$ and $k \in K$ the following inequality holds:*

$$|\mathcal{Y}_i w|(n, r, k) \le C_\varepsilon \sum_{\nu=1}^{q} \Big\{ e^{r(\rho_\nu + \varepsilon)} w(n)^{\frac{Q - \rho_\nu - \varepsilon}{Q}} + e^{r(\rho_\nu - \varepsilon)} w(n)^{\frac{Q - \rho_\nu + \varepsilon}{Q}} \Big\}\,. \tag{11}$$

If D is semisimple, one can estimate

$$|\mathcal{Y}_i w|(n, r, k) \le C_0 \sum_{\nu=1}^{q} e^{r\rho_\nu} w(n)^{\frac{Q - \rho_\nu}{Q}}\,, \tag{12}$$

with a constant $C_0 > 0$ independent of i, n, r and k.

Proof. For $i = -n, ..., -1$ the functions $\mathcal{X}_i w$ are identically zero, as w does not depend on the variable k.

Using formula (10), we get for $n \in F_0$ and $i = \{1, ..., m\}$

$$\begin{aligned} &[(e^{rD}\mathcal{X}_i)^N w](e^{sD}n) = e^{sQ} \frac{d}{dt} w(n \cdot \exp_N(t e^{(r-s)D}\mathcal{X}_i))|_{t=0} \\ =& e^{sQ}(e^{(r-s)D}\mathcal{X}_i)^N w(n) = \sum_{\nu=1}^{q} e^{sQ + (r-s)\rho_\nu} \sum_{l=1}^{m} P_{i,l,\nu}(r-s)(\mathcal{X}_l^N w)(n)\,. \end{aligned}$$

Hence there exists for any $\varepsilon > 0$ a constant $c_\varepsilon > 0$, fulfilling

$$|(e^{rD}\mathcal{X}_i)^N w|(e^{sD}n) \le c_\varepsilon \sum_{\nu=1}^{q} e^{sQ + (r-s)\rho_\nu + |r-s|\varepsilon} \sum_{l=1}^{m} |\mathcal{X}_l^N w|(n)\,. \tag{13}$$

Now define $C := c_\varepsilon \max\{\sum_{l=1}^{N} |(\mathcal{X}_l^N w)|(n) : n \in F_0\}$. Because of $w|_{F_0} \equiv 1$, there holds for $n \in F_0$, $r \in \mathbb{R}$

$$\begin{aligned} &|(e^{rD}\mathcal{X}_i)^N w|(e^{sD}n) \\ \le& C e^{sQ} \sum_{\nu=1}^{q} \{ e^{(r-s)(\rho_\nu + \varepsilon)} w(n)^{\frac{Q - \rho_\nu - \varepsilon}{Q}} + e^{(r-s)(\rho_\nu - \varepsilon)} w(n)^{\frac{Q - \rho_\nu + \varepsilon}{Q}} \} \\ =& C \sum_{\nu=1}^{q} \{ e^{r(\rho_\nu + \varepsilon)} w(e^{sD}n)^{\frac{Q - \rho_\nu - \varepsilon}{Q}} + e^{r(\rho_\nu - \varepsilon)} w(e^{sD}n)^{\frac{Q - \rho_\nu + \varepsilon}{Q}} \}\,. \end{aligned} \tag{14}$$

We have $N \setminus \{0\} = \{e^{sD}F_0 : s \in \mathbb{R}\}$. Furthermore, the functions $\tilde{\gamma}_{i,j}$ are bounded on K and the $\mathcal{Y}_i$s are linear combinations of the $\mathcal{X}_j s$. Thus formulae (9) and (14) imply inequality (11). If D is semisimple, (13) can be simplified to

$$|(e^{rD}\mathcal{X}_i)^N w|(e^{sD}n) \leq c_0 \sum_{\nu=1}^{q} e^{sQ+(r-s)\rho_\nu} \sum_{l=1}^{m} |\mathcal{X}_l^N w|(n)$$

with a suitable constant $c_0 > 0$. Therefore inequality (12) follows. □

To simplify the proof of Proposition 2, we state two preliminary lemmas:

Lemma 3. *For any $\delta > 0$ there exists a $C > 0$ such that for each $j \in \{1, ..., p\}$ and each $z \in \mathbb{C}$ with $\Re(z) \geq \delta$ the inequality $\|e^{r\frac{Q}{2}}\mathcal{Y}_j\phi_z\| \leq C$ holds; here $e^{r\frac{Q}{2}}$ denotes the function $(n, r, k) \mapsto e^{r\frac{Q}{2}}$.*

Proof. Let $z \in \mathbb{C}$ with $\Re(z) \geq \delta$. If $\mathcal{Y}_j$ has the form $\mathcal{Y}_j = \sum_{i=-n}^{m} a_{i,j}\mathcal{X}_i$ with $a_{i,j} \in \mathbb{R}$, then, by using the notation $\eta_j := \sum_{i=0}^{m} a_{i,j}\tilde{\gamma}_{0,i}$, we get

$$\langle \Delta\phi_z, e^{rQ}\phi_z\rangle = \sum_{j=1}^{p}(\|e^{r\frac{Q}{2}}\mathcal{Y}_j\phi_z\|^2 + \langle \mathcal{Y}_j\phi_z, \eta_j Q e^{rQ}\phi_z\rangle)\,.$$

The Cauchy-Schwarz inequality and the fact that ϕ_z solves the homogeneous heat equation with respect to Δ imply

$$\sum_{j=1}^{p} \|e^{r\frac{Q}{2}}\mathcal{Y}_j\phi_z\|^2 \leq |\langle e^{r\frac{Q}{2}}\Delta\phi_z, e^{r\frac{Q}{2}}\phi_z\rangle| + \sum_{j=1}^{p} |\eta_j|_\infty Q\langle |e^{r\frac{Q}{2}}\mathcal{Y}_j\phi_z|, |e^{r\frac{Q}{2}}\phi_z|\rangle$$

$$\leq \|e^{r\frac{Q}{2}}\partial_z\phi_z\|\|e^{r\frac{Q}{2}}\phi_z\| + \sum_{j=1}^{p}\frac{1}{2}\left(\|e^{r\frac{Q}{2}}\mathcal{Y}_j\phi_z\|^2 + |\eta_j|_\infty^2 Q^2\|e^{r\frac{Q}{2}}\phi_z\|^2\right).$$

From $(e^{-z\Delta})^* = e^{-z^*\Delta}$ follows $\phi_z(g^{-1}) = m(g)\phi_z(g)$. As the modular function m is given by e^{rQ}, we get $\|e^{r\frac{Q}{2}}\phi_z\| = \|\phi_z\|$. Since $\phi_z = e^{-(z-\delta)\Delta}\phi_\delta$, $\|e^{r\frac{Q}{2}}\phi_z\| \leq \|\phi_\delta\|$ holds. By using Cauchy's formula, it is easy to verify that

$$\|e^{r\frac{Q}{2}}\partial_z\phi_z\| \leq \frac{2}{\delta}\sup\{\|e^{r\frac{Q}{2}}\phi_\zeta\| : |z-\zeta| < \tfrac{\delta}{2}\}\,.$$

□

Lemma 4. *Let $j \in \{1, ..., p\}$, $\delta > 0$, $\eta > 0$ and $\tilde{C} > 0$.*

(i) If $Q \geq 2\eta$, there exists a $C > 0$ such that for $\alpha > 0$ and z with $\Re(z) \geq \delta$

$$\langle |\mathcal{Y}_j\phi_z|, e^{r\eta}w^{\frac{Q-\eta}{Q}}|\phi_z|\rangle \leq \frac{\alpha}{\tilde{C}}\|w^{1/2}\mathcal{Y}_j\phi_z\|^2 + \frac{C}{\alpha}\|w^{1/2}\phi_z\|^{\frac{2Q-4\eta}{Q}}\,. \tag{15}$$

(ii) If $Q < 2\eta < 2Q$, there exists a $C > 0$ with

$$\langle |\mathcal{Y}_j\phi_z|, e^{r\eta}w^{\frac{Q-\eta}{Q}}|\phi_z|\rangle \leq \frac{\alpha}{\tilde{C}}\|w^{1/2}\mathcal{Y}_j\phi_z\|^2 + C\alpha^{\frac{\eta-Q}{\eta}} \tag{16}$$

for all $\alpha > 0$ and z with $\Re(z) \geq \delta$.

Proof. Consider α, $\tilde{C}$, η, $\delta > 0$. Let $j \in \{1, ..., p\}$ and z with $\Re(z) \geq \delta$.

(i) $Q \geq 2\eta$ implies

$$\langle |\mathcal{Y}_j \phi_z|, e^{r\eta} w^{\frac{Q-\eta}{Q}} |\phi_z| \rangle \leq \frac{\alpha}{\tilde{C}} \|w^{1/2} \mathcal{Y}_j \phi_z\|^2 + \frac{\tilde{C}}{\alpha} \|e^{r\eta} w^{\frac{Q-2\eta}{2Q}} \phi_z\|^2 .$$

By using Hölder's inequality, one gets

$$\|e^{r\eta} w^{\frac{Q-2\eta}{2Q}} \phi_z\| \leq \|e^{r\frac{Q}{2}} \phi_z\|^{\frac{2\eta}{Q}} \|w^{1/2} \phi_z\|^{\frac{Q-2\eta}{Q}} \leq \|\phi_\delta\|^{\frac{2\eta}{Q}} \|w^{1/2} \phi_z\|^{\frac{Q-2\eta}{Q}} .$$

(ii) Let $Q < 2\eta < 2Q$. By using Hölder's inequality with exponents $p = Q/2(Q-\eta)$ and $p' = Q/(2\eta - Q)$, we can estimate

$$\begin{aligned} &\langle |\mathcal{Y}_j \phi_z|, e^{r\eta} w^{\frac{Q-\eta}{Q}} |\phi_z| \rangle \leq \|e^{r\frac{Q}{2}} \phi_z\| \|e^{r(\eta - \frac{Q}{2})} w^{\frac{Q-\eta}{Q}} \mathcal{Y}_j \phi_z\| \\ \leq &\|\phi_\delta\| \|e^{r(\eta-\frac{Q}{2})} w^{\frac{Q-\eta}{Q}} \mathcal{Y}_j \phi_z\| \leq \|\phi_\delta\| \|w^{1/2} \mathcal{Y}_j \phi_z\|^{\frac{2(Q-\eta)}{Q}} \|e^{\frac{rQ}{2}} \mathcal{Y}_j \phi_z\|^{\frac{2\eta - Q}{Q}} . \end{aligned}$$

As there exists a constant $C > 0$ with $\|e^{r\frac{Q}{2}} \mathcal{Y}_j \phi_z\| \leq C$ (cf. Lemma 3) and as the inequality $ab \leq a^r + b^{r'}$, $r' = \frac{r}{r-1}$ holds for all a, $b > 0$ and $r > 1$, we obtain inequality (16) with $r = Q/(Q - \eta)$ and a suitable $C > 0$. □

Proof of Proposition 2. Lemma 1 states the existence of a $C > 0$ with $w(g) \leq Ce^{Cd(g)}$, and $(e^{-z\Delta})_{\Re(z)>0}$ is a holomorphic semigroup of operators on each weighted L^2-space $L^2(G, e^{sd(g)}\, d^r g)$, $s \in \mathbb{R}$ (cf. [5], Lemma 1.2). Hence $z \mapsto w^{1/2}\phi_z$ is a holomorphic mapping from the right open complex halfplane into $L^2(G)$. Therefore there exists a $C > 0$ such that $\|w^{1/2}\phi_{1+is}\| \leq C$ for each $s \in [0, 1]$. Since $\phi_{1-is} = (\phi_{1+is})^*$, we have to consider only the case where $s \geq 1$. For $0 < \alpha \leq 1$ we define

$$\psi_\alpha(s) := \|w^{1/2} \phi_{\frac{1}{2}+(i+\alpha)s}\|^2 .$$

Further we define $z := \frac{1}{2} + (i + \alpha)s$. Using this notation, we obtain

$$\begin{aligned} \partial_s \psi_\alpha(s) &= 2\Re\langle (i+\alpha)\partial_z \phi_z, w\phi_z \rangle = -2\Re\big((i+\alpha)\langle \Delta\phi_z, w\phi_z\rangle\big) \\ &\leq 2 \sum_{j=1}^{p} \big(-\alpha \|w^{1/2} \mathcal{Y}_j \phi_z\|^2 + 2\langle |\mathcal{Y}_j \phi_z|, |\mathcal{Y}_j w| |\phi_z| \rangle \big) . \end{aligned} \tag{17}$$

Case (1): $\dim \mathfrak{n} = 1$. Here, D is given by the 1×1-matrix (Q). Estimate (12) leads us to the inequality $|\mathcal{Y}_j w|(n, r, k) \leq C_0 e^{rQ}$ for $j \in \{1, ..., p\}$. Hence

$$\langle |\mathcal{Y}_j \phi_z|, |\mathcal{Y}_j w| |\phi_z| \rangle \leq C_0 \|e^{r\frac{Q}{2}} \mathcal{Y}_j \phi_z\| \|e^{r\frac{Q}{2}} \phi_z\| \leq C ,$$

with $C > 0$ independent of s, α (cf. Lemma 3). Thus there exists a $C > 0$ such that $\partial_s \psi_\alpha(s) \leq C$ for all s and α. That implies (with $\alpha := \frac{1}{2s}$)

$$\|w^{1/2} \phi_{1+is}\| \leq C\sqrt{1 + |s|} \quad \text{for } s \geq 1.$$

Case (2): $\dim \mathfrak{n} \geq 2$. For $j = 1, ..., p$ and $\varepsilon > 0$ Proposition 3 implies

$$\begin{aligned} &\langle |\mathcal{Y}_j \phi_z|, |\mathcal{Y}_j w||\phi_z|\rangle \\ \leq & C_\varepsilon \sum_{\nu=1}^{q} \langle |\mathcal{Y}_j \phi_z|, (e^{r(\rho_\nu+\varepsilon)} w^{\frac{Q-(\rho_\nu+\varepsilon)}{Q}} + e^{r(\rho_\nu-\epsilon)} w^{\frac{Q-(\rho_\nu-\varepsilon)}{Q}})|\phi_z|\rangle \,. \end{aligned} \tag{18}$$

(If D is semisimple, we can exchange ε by 0 in (18) and in the rest of this proof.) For every $\varepsilon < \rho$, $\rho_\nu + \varepsilon$ fulfills $Q > \rho_\nu + \varepsilon$ for any $\nu \in \{1, ..., q\}$. According to (17), (18), (15) and (16) (with $\tilde{C} := 4qC_\varepsilon$), $\partial_s \psi_\alpha$ is majorized by a sum over $\eta \in \{\rho_\nu \pm \varepsilon : \nu = 1, ..., q\}$, consisting of terms of the form

$$\frac{C}{\alpha} \|w^{1/2}\phi_z\|^{\frac{2Q-4\eta}{Q}} = \frac{C}{\alpha}\, \psi_\alpha^{\frac{Q-2\eta}{Q}} \qquad \text{for } Q \geq 2\eta$$

and

$$C\alpha^{\frac{\eta-Q}{\eta}} \leq \frac{C}{\alpha} \qquad \text{for } Q < 2\eta < 2Q.$$

Hence, there exists a $C > 0$ such that for all $\alpha \in]0, 1]$, $s \geq 1$ the function ψ_α is majorized by the solution u of the initial value problem

$$u'(s) = \frac{C}{\alpha}(1 + u(s))^{\frac{Q-2(\rho-\varepsilon)}{Q}}\,, \quad u(1) = \psi_\alpha(1)\,,$$

which is given by

$$u(s) = \Big(\frac{2(\rho-\varepsilon)C}{Q\alpha}(s-1) + (1+\psi_\alpha(1))^{\frac{2(\rho-\varepsilon)}{Q}}\Big)^{\frac{Q}{2(\rho-\varepsilon)}} - 1\,.$$

Hence, for $\alpha = \frac{1}{2s}$ there exists a constant $c_\varepsilon > 0$, independent of s, with

$$\|w^{1/2}\phi_{1+is}\| = \sqrt{\psi_\alpha(s)} \leq c_\varepsilon (1 + |s|)^{\frac{Q}{2(\rho-\varepsilon)}}\,.$$

□

3 Semidirect products of the Heisenberg group $\mathbb{H}_1$ and the real axis

3.1 Motivation

Noteworthy about Theorem 1 and 2 is, that the exponents in (2), (3) and the exponent κ in Theorem 2 tend to infinity with the ratio Q/ρ. There are indications that this phenomenon might be a consequence of our method of proof and does not reflect any underlying mathematical reality.

In [10] e. g., groups of the form $H = \mathbb{R}^2 \rtimes_\theta \mathbb{R}$ with $\theta(t) = e^{tD}$, D any 2×2-matrix, were studied and for distinguished sub-Laplacians and their heat kernels the estimate

$$\|\phi_{1+i\xi}\|_{L^1(H)} \leq C(1 + |\xi|)^5 \tag{19}$$

was proven. (In some cases the estimates are better; the method which handled the most delicate case had been introduced in [5].) So the exponent of (19) is bounded, regardless of the action e^{tD}.

If we consider semidirect products $\mathbb{H}_1 \rtimes_\theta \mathbb{R}$ of the 3-dimensional Heisenberg group with the real axis, we can of course not expect such a result. The article [2] shows that not even all of this semidirect products admit differentiable L^p-functional calculus.

But if we confine ourselves to group homomorphisms θ, which are induced by derivations D in diagonal form with non-negative entries (or non-positive entries), we are able to derive an estimate like (19) with exponent 6 for all θ by transferring the methods from [5] and [10] to our situation.

3.2 Preliminaries

The *Heisenberg group* $\mathbb{H}_1$ is the set $\mathbb{R}^3$ endowed with the multiplication

$$(x, y, u)(x', y', u') = \Big(x + x', y + y', u + u' + \frac{1}{2}(xy' - yx')\Big)\,.$$

The Lie algebra of $\mathbb{H}_1$ is the *Heisenberg algebra* $\mathfrak{h}_1$.

Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta \geq 0$, and let D be the derivation on $\mathfrak{h}_1$ defined by $D(p, q, t) = (\alpha p, \beta q, (\alpha+\beta)t)$. The trace Q of D is then equal to $2(\alpha+\beta)$. Here our object of interest is the group $G := \mathbb{H}_1 \rtimes_\theta \mathbb{R}$, where $\theta(r) = e^{rD}$.

As in Section 2 the right invariant Haar measure $d^r g$ is simply the Lebesgue measure $dx\,dy\,du\,dr$ on $\mathbb{R}^4$ and the modular function is $m(g) = m(x, y, u, r) = e^{2(\alpha+\beta)r}$.

The left invariant vector fields on G, induced by the Lie algebra elements $\mathcal{X}_1 := (1, 0, 0, 0)$, $\mathcal{X}_2 := (0, 1, 0, 0)$, $\mathcal{X}_3 := (0, 0, 1, 0)$ and $\mathcal{X}_0 := (0, 0, 0, 1)$ from $\mathfrak{g} = \mathfrak{h}_1 \times \mathbb{R}$, are explicitly given by

$$\begin{aligned} \mathcal{X}_1 &= e^{\alpha r}\Big(\partial_x - \frac{1}{2}y\partial_u\Big)\,, \quad & \mathcal{X}_2 &= e^{\beta r}\Big(\partial_y + \frac{1}{2}x\partial_u\Big)\,,\\ \mathcal{X}_3 &= e^{(\alpha+\beta)r}\partial_u\,, & \mathcal{X}_0 &= \partial_r\,. \end{aligned}$$

The operator $\Delta_S := -\sum_{j=0}^{2} \mathcal{X}_j^2$ is a sub-Laplacian and $\Delta_L := -\sum_{j=0}^{3} \mathcal{X}_j^2$ a full Laplacian on G. Let ϕ_t^S and ϕ_t^L denote the heat kernels of Δ_S and Δ_L, respectively. Further let $J(\Delta_S) := \{0, 1, 2\}$ and $J(\Delta_L) := \{0, 1, 2, 3\}$. In the sequel Δ shall denote the sub-Laplacian Δ_S as well as the full Laplacian Δ_L, and $(\phi_t)_{t>0}$ shall denote the heat kernel of Δ.

3.3 Results

Theorem 3. *There exists a $C > 0$ such that for each $\xi \in \mathbb{R}$ the inequality*

$$\|\phi_{1+i\xi}\|_{L^1(G)} \leq C(1 + |\xi|)^\kappa \tag{20}$$

holds; hereby we have

$$\kappa = \begin{cases} \frac{\alpha+\beta}{\min\{\alpha,\beta\}} + 2 & \text{if } \frac{\alpha}{\beta} \in [\frac{1}{3}, 3], \\ 6 & \text{otherwise.} \end{cases} \tag{21}$$

Like Theorem 1, Theorem 3 implies a multiplier theorem:

Theorem 4. *Let $p \in [1,\infty]$, $\varepsilon > 0$ and κ as in (21). Then each $f \in C_c \cap H^{\kappa+\frac{1}{2}+\varepsilon}(\mathbb{R})$ is an L^P-multiplier for Δ.*

Remark 2. (a) Extending the results of Theorem 3 and Theorem 4 to compact extensions $G \rtimes_\gamma K$ of G and sub-Laplacians $\Delta_K + d\gamma(\Delta)$, Δ_K a sub-Laplacian on K, is more or less trivial:

If $\alpha \neq \beta$, it can be shown that any homomorphism $\gamma : K \to \mathrm{Aut(G)}$ has to be trivial, i. e., $\gamma(k)$ is the identity on G for any $k \in K$. (One can e. g. calculate the entries of the matrix $\tilde{\gamma} = d\gamma$ successively.) But then our sub-Laplacian is of the form $\Delta_K + \Delta$ and its heat kernel is given by $\phi_z^K \otimes \phi_z$, ϕ_z^K the heat kernel of Δ_K.

If $\alpha = \beta \neq 0$, the extension of the results is contained in Section 2.

If $\alpha = 0 = \beta$, then $G \rtimes_\gamma K$ has polynomial growth, so we refer to [1].

(b) If $\phi_z = \phi_z^S$ is the heat kernel of the sub-Laplacian and if $\alpha = \beta \neq 0$, Inequality (20) and Theorem 4 hold even with $\kappa = 3/2$ (cf. [9] or [4]).

3.4 Outline of the proof of Theorem 3

The general strategy for proving Theorem 3 is the same as in the proof of Theorem 1. That is, we want to reduce the L^1-estimate of the heat kernel to a weighted L^2-estimate. But this time we utilize a weight function w, which is independent of the action θ: We define $w : G \to \mathbb{R}$ by

$$w(x, y, u, r) = (1 + |x|)(1 + |y|)(1 + |u|) .$$

Then there exists a $C > 0$ such that for any $R > 0$

$$\int_{d(g)\leq R} w(g)^{-1}\, d^r g \leq C(1 + R)^4 .$$

Again, by using the same argument as in [5], we are able to find a constant $C > 0$ such that for each $\xi \in \mathbb{R}$

$$\|\phi_{1+i\xi}\|_{L^1(G)} \leq C(1 + |\xi|)^4 (1 + \|w^{1/2}\phi_{1+i\xi}\|) . \tag{22}$$

Therefore we are interested in estimating the term $|\partial_\xi \|w^{1/2}\phi_{1+i\xi}\|^2|$. We will do this step by step, beginning with weights of low order in x, y, u (like $|x|^{1/2}$ and $|y|^{1/2}$) and using the estimations of these terms to estimate higher order terms in x, y and u. We start with the analogue of Lemma 3:

Lemma 5. *For any $\delta > 0$ there exists a $C > 0$ such that for each $j \in J(\Delta)$ and each $z \in \mathbb{C}$ with $\Re(z) \geq \delta$ the inequality $\|e^{(\alpha+\beta)r}\mathcal{X}_j\phi_z\| \leq C$ holds.*

Lemma 5 can be verified easily by mimicking the proof of Lemma 3.

Lemma 6. *For any compact set* $K \subset]0,\infty[$ *there exists a* $C > 0$ *with*

$$\| \, |x|^{1/2} \phi_{\rho+i\xi} \| + \| \, |y|^{1/2} \phi_{\rho+i\xi} \| \le C(1+|\xi|)^{1/2} \tag{23}$$

and

$$\| \, |x|^{1/2} \mathcal{X}_j \phi_{\rho+i\xi} \| + \| \, |y|^{1/2} \mathcal{X}_j \phi_{\rho+i\xi} \| \le C(1+|\xi|)^{1/2} \tag{24}$$

for all $\rho \in K$, $\xi \in \mathbb{R}$ *and all* $j \in J(\Delta)$.

Proof. By using the notation $z = \rho + i\xi$ we get

$$\langle \Delta \phi_z, |y| \phi_z \rangle = \sum_{j \in J(\Delta)} \| \, |y|^{1/2} \mathcal{X}_j \phi_z \|^2 + \langle \mathcal{X}_2 \phi_z, e^{\beta r} \operatorname{sgn}(y) \phi_z \rangle \, .$$

On the one hand, from this and Lemma 5 there follows

$$\begin{aligned} |\partial_\xi \| \, |y|^{1/2} \phi_z \|^2| &= 2|\Re(i\langle \partial_z \phi_z, |y|\phi_z \rangle)| = 2|\Im \langle \Delta \phi_z, |y| \phi_z \rangle| \\ &\le 2|\langle \mathcal{X}_2 \phi_z, e^{\beta r} \operatorname{sgn}(y) \phi_z \rangle| \le 2\|\phi_z\| \, \|e^{\beta r} \mathcal{X}_2 \phi_z\| \le C \, . \end{aligned}$$

We obtain $\| \, |y|^{1/2} \phi_z \|^2 \le C(1+|\xi|)$, because the mapping $K \ni \rho \mapsto \| \, |y|^{1/2} \phi_\rho \|^2$ is in particular continuous and thus bounded.

On the other hand, it follows from $(\partial_z + \Delta)\phi_z = 0$ and Cauchy's formula that

$$\sum_{j \in J(\Delta)} \| \, |y|^{1/2} \mathcal{X}_j \phi_z \|^2 \le C + \| \, |y|^{1/2} \phi_z \| \, \| \, |y|^{1/2} \partial_z \phi_z \| \le C(1+|\xi|) \, .$$

The rest of the statement can be obtained analogously. □

Lemma 7. *For any compact set* $K \subset]0,\infty[$ *one can choose a* $C > 0$ *in such a way that for each* $\rho \in K$, $\xi \in \mathbb{R}$ *and each* $j \in J(\Delta)$

$$\|e^{(\frac{\alpha}{2}+\beta)r} |x|^{1/2} \phi_{\rho+i\xi}\| + \|e^{(\alpha+\frac{\beta}{2})r} |y|^{1/2} \phi_{\rho+i\xi}\| \le C(1+|\xi|)^{1/2} \tag{25}$$

and

$$\|e^{(\frac{\alpha}{2}+\beta)r} |x|^{1/2} \mathcal{X}_j \phi_{\rho+i\xi}\| + \|e^{(\alpha+\frac{\beta}{2})r} |y|^{1/2} \mathcal{X}_j \phi_{\rho+i\xi}\| \le C(1+|\xi|)^{1/2} \, . \tag{26}$$

Proof. Let again $z := \rho + i\xi$. With the notation $\gamma(r) = \exp((\alpha + \frac{\beta}{2})r)$ we get $\|\gamma |y|^{1/2} \phi_{\rho+i\xi}\| = \| |y|^{1/2} \phi_{\rho+i\xi}\|$, because of $\phi_z(g^{-1}) = m(g)\phi_z(g)$. Further

$$\begin{aligned} \langle \Delta \phi_z, \gamma^2 |y| \phi_z \rangle = &\sum_{j \in J(\Delta)} \|\gamma |y|^{1/2} \mathcal{X}_j \phi_z\|^2 + \langle \mathcal{X}_2 \phi_z, m \operatorname{sgn}(y) \phi_z \rangle \\ &+ \langle \mathcal{X}_0 \phi_z, (2\alpha+\beta)\gamma^2 |y| \phi_z \rangle \, . \end{aligned}$$

Here the absolute value of the last term can be majorized by

$$\frac{1}{2} \|\gamma |y|^{1/2} \mathcal{X}_0 \phi_z\|^2 + \frac{(2\alpha+\beta)^2}{2} \|\gamma |y|^{1/2} \phi_z\|^2 \, .$$

Hence

$$\sum_{j\in J(\Delta)} \|\gamma|y|^{1/2}\mathcal{X}_j\phi_z\|^2 \leq 2\|\gamma|y|^{1/2}\phi_z\|\,\|\gamma|y|^{1/2}\partial_z\phi_z\| + 2\|\sqrt{m}\phi_z\|\,\|\sqrt{m}\mathcal{X}_2\phi_z\| + (2\alpha+\beta)^2\|\gamma|y|^{1/2}\phi_z\|^2 \leq C(1+|\xi|)\,.$$

□

Lemma 8. *For any compact set $K\subset]0,\infty[$ there exists a $C>0$ with*

$$\|\,|u|^{1/2}\phi_{\rho+i\xi}\| + \|\,|u|^{1/2}\mathcal{X}_j\phi_{\rho+i\xi}\| \leq C(1+|\xi|) \tag{27}$$

for every $\rho\in K$, $\xi\in\mathbb{R}$ and $j\in J(\Delta)$.

Proof. We consider just the case $\Delta=\Delta_L$. (The proof for the heat kernel of the sub-Laplacian Δ_S is contained in the proof for the Laplacian Δ_L – one has just to ignore all terms in which $\mathcal{X}_3$ occurs.)

With $z=\rho+i\xi$ we get

$$\langle\Delta_L\phi_z^L,|u|\phi_z^L\rangle = \sum_{j=0}^{3}\|\,|u|^{1/2}\mathcal{X}_j\phi_z^L\|^2 - \frac{1}{2}\langle\mathcal{X}_1\phi_z^L, e^{\alpha r}y\,\mathrm{sgn}(u)\phi_z^L\rangle + \frac{1}{2}\langle\mathcal{X}_2\phi_z^L, e^{\beta r}x\,\mathrm{sgn}(u)\phi_z^L\rangle + \langle\mathcal{X}_3\phi_z^L, e^{(\alpha+\beta)r}\,\mathrm{sgn}(u)\phi_z^L\rangle\,.$$

By proceeding as in the proof of (23) we obtain $|\partial_\xi\|\,|u|^{1/2}\phi_z^L\|^2|\leq C(1+|\xi|)$, and from this follows $\|\,|u|^{1/2}\phi_{\rho+i\xi}^L\|^2\leq C(1+|\xi|)^2$. As in the proof of (24) one derives eventually $\sum_{j=0}^3\|\,|u|^{1/2}\mathcal{X}_j\phi_z^L\|^2\leq C(1+|\xi|)^2$. □

In Lemma 7 we got the same upper bound $C(1+|\xi|^{1/2})$ for the terms $\|e^{(\alpha+\frac{\beta}{2})r}|y|^{1/2}\phi_{\rho+i\xi}\|$ and $\|e^{(\alpha+\frac{\beta}{2})r}|y|^{1/2}\mathcal{X}_j\phi_{\rho+i\xi}\|$ (which we shall call *related terms of* $\||y|^{1/2}\phi_{\rho+i\xi}\|$) as for the terms $\||y|^{1/2}\phi_{\rho+i\xi}\|$ and $\||y|^{1/2}\mathcal{X}_j\phi_{\rho+i\xi}\|$ in Lemma 6. Similarly, we get from Lemma 8 the estimate

$$\|e^{\frac{\alpha+\beta}{2}r}|u|^{1/2}\phi_{\rho+i\xi}\| + \|e^{\frac{\alpha+\beta}{2}r}|u|^{1/2}\mathcal{X}_j\phi_{\rho+i\xi}\| \leq C(1+|\xi|) \tag{28}$$

for the sum of the related terms of $\||u|^{1/2}\phi_{\rho+i\xi}\|$. The last inequality holds again for any compact $K\subset]0,\infty[$, $\rho\in K$, $\xi\in\mathbb{R}$, $j\in J$ and the constant C depends just on K.

By using the same notation and just the same techniques that we have used so far, we obtain

$$\|x\phi_{\rho+i\xi}\| + \|y\phi_{\rho+i\xi}\| \leq C(1+|\xi|) \tag{29}$$

and this upper bound holds also for the related terms $\|e^{\beta r}x\phi_{\rho+i\xi}\|$, $\|e^{\alpha r}y\phi_{\rho+i\xi}\|$, $\|e^{\beta r}x\mathcal{X}_j\phi_{\rho+i\xi}\|$ and $\|e^{\alpha r}y\mathcal{X}_j\phi_{\rho+i\xi}\|$.

After this it is easy to derive that $\||xu|^{1/2}\phi_{\rho+i\xi}\|$, $\||yu|^{1/2}\phi_{\rho+i\xi}\|$ and the related terms are bounded by $C(1+|\xi|)^{3/2}$.

In a similar way one establishes

$$\|x|y|^{1/2}\phi_{\rho+i\xi}\| + \||x|^{1/2}y\phi_{\rho+i\xi}\| \leq C(1+|\xi|)^{3/2} \tag{30}$$

(and this bound holds also for the related terms). With this bunch of estimates we are able to verify $\|w^{1/2}\phi_{1+i\xi}\| \leq C(1+|\xi|)^2$. Therefore, with regard to inequality (22) and Theorem 1, Theorem 3 holds.

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References

1. G. Alexopoulos, Spectral multipliers on Lie groups of polynomial growth, Proc. AMS **120** (1994), 973 - 979.
2. M. Christ, D. Müller, On L^p spectral multipliers for a solvable Lie group, Geom. Funct. Anal. **6** (1996), 860 - 876.
3. M. G. Cowling, S. Giulini, A. Hulanicki, G. Mauceri, Spectral multipliers for a distinguished Laplacian on certain groups of exponential growth, Studia Math. **111** (1994), 103 - 121.
4. M. Gnewuch, Zum differenzierbaren L^p–Funktionalkalkül auf Lie–Gruppen mit exponentiellem Volumenwachstum (Dissertation thesis, Kiel 2002), 157 p.
5. W. Hebisch, Boundedness of L^1 spectral multipliers for an exponential solvable Lie group, Colloq. Math. **73** (1997), 155 - 164.
6. W. Hebisch, Spectral multipliers on exponential growth solvable Lie groups, Math. Zeitschr. 229 (1998), 435 - 441.
7. W. Hebisch, J. Ludwig, D. Müller, Sub-Laplacians of holomorphic L^p-type on exponential solvable groups, Preprint (Berichtsreihe des Mathematischen Seminars Kiel 01-3, April 2001).
8. J. Ludwig, D. Müller, Sub-Laplacians of holomorphic L^p-type on rank one AN-groups and related solvable groups, J. of Funct. Anal. **170** (2000), 366 - 427.
9. S. Mustapha, Multiplicateurs de Mikhlin pour une classe particulière de groupes non-unimodulaires, Ann. Inst. Fourier **48** (1998), 957 - 966.
10. S. Mustapha, Le problème de Mikhlin sur les groupes de Lie résolubles, In: Habilitation thesis, 1998.
11. N. T. Varopoulos, L. Saloff-Coste, T. Coulhon, Analysis and Geometry on Groups, Cambrigde University Press, 1992.